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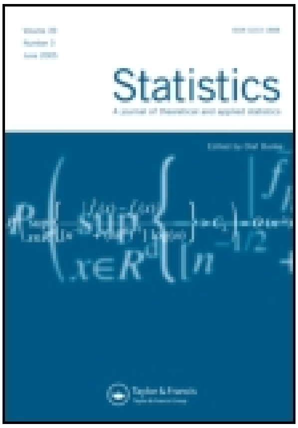

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Characterizations of geometric distribution through progressively Type-II right-censored order statistics

N. Balakrishnan [a], E. Cramer [b] & A. Dembińska [c]

[a] Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, ON, Canada, L8S 4K1

[b] Institute of Statistics, RWTH Aachen University, 52056, Aachen, Germany

[c] Faculty of Mathematics and Information Science, Warsaw University of Technology, Plac. Politechniki 1, 00-661, Warsaw, Poland

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Characterizations of geometric distribution through progressively Type-II right-censored order statistics

N. Balakrishnan[a], E. Cramer[b]* and A. Dembińska[c]

[a]*Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, ON, Canada L8S 4K1;* [b]*Institute of Statistics, RWTH Aachen University, 52056 Aachen, Germany;* [c]*Faculty of Mathematics and Information Science, Warsaw University of Technology, Plac. Politechniki 1, 00-661 Warsaw, Poland*

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In this paper, we consider characterizations of geometric distribution based on some properties of progressively Type-II right-censored order statistics. Specifically, we establish characterizations through conditional expectation, identical distribution, and independence of functions of progressively Type-II right-censored order statistics. Moreover, extensions of these results to generalized order statistics are also sketched. These generalize the corresponding results known for the case of ordinary order statistics.

Keywords: progressively Type-II right-censored order statistics; generalized order statistics; characterizations; geometric distribution

AMS 2000 Subject Classification: 62 G10; 62 G32

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1. Introduction

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables (rv's) with cumulative distribution function (cdf) F. By arranging these rvs in non-decreasing order of magnitude, we obtain order statistics, denoted by $X_{1:n} \leq \cdots \leq X_{n:n}$. In some situations, during an experiment, not all X_is are observed as there may be censoring involved with only the corresponding censored order statistics being observed. In this paper, we consider the so-called progressive Type-II right censoring in which only m out of n rvs are observed and the remaining $n - m$ rvs are censored progressively from the experiment. More specifically, the smallest observation $X_{1:n}$ becomes the first progressively Type-II right-censored order statistic observed, then we remove at random R_1 rvs from the remaining $n - 1$ rvs, and the smallest observation among the remaining $n - R_1 - 1$ becomes the second progressively Type-II right-censored order statistic observed, and so on. This procedure is continued so that after observing the ith ($i = 1, \dots, m - 1$) progressively Type-II right-censored order statistic, R_i rvs are withdrawn from the remaining rvs, and the smallest observation among the remaining $n - R_1 - R_2 - \cdots - R_i - i$ rvs becomes the

*Corresponding author. Email: erhard.cramer@rwth-aachen.de

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$(i+1)$th progressively Type-II right-censored order statistic. The experiment terminates when the mth progressively Type-II right-censored order statistic is observed with all the remaining $n - R_1 - \cdots - R_{m-1} - m = R_m$ rvs are being censored. The censored sample so observed is denoted by $(X_{1:m:n}^{(R_1,\ldots,R_m)}, X_{2:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)})$ and is referred to as progressively Type-II right-censored order statistics with censoring scheme $(R_1, R_2, \ldots, R_m)$.

Distributional properties and inferential procedures based on progressively Type-II right-censored order statistics have been studied rather extensively in the literature; see, for example, the book by Balakrishnan and Aggarwala [1] and the recent article by Balakrishnan [2] for a detailed review of developments on this area of research. However, almost all the work focuses on the case when the underlying cdf F is continuous. If F is discrete, the probability of ties among observations is positive and so at the first glance the study of progressively Type-II right-censored order statistics appears to be very complicated. But, it became feasible when Balakrishnan and Dembińska [3] noticed that the distribution of progressively Type-II right-censored order statistics from any cdf F can be expressed by the well-known distribution of progressively censored sample from the standard uniform cdf. Their observation specifically implies that progressively Type-II right-censored order statistics from an arbitrary distribution can be embedded in the model of generalized order statistics introduced by Kamps [4,5] (see also [6]). This allows for the theory of discrete generalized order statistics developed by Tran [7] to be utilized for establishing distributional properties of discrete progressively Type-II right-censored samples (see [8] for details). In particular, by making use of Tran's results, one can obtain the following characterizations of geometric, modified geometric, and geometric-type distributions.

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Theorem 1 [7] *Let $X_{1:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ be progressively Type-II right-censored order statistics from a discrete cdf F with support S.*

(1) *If $S = \{0, 1, \ldots\}$, then the rv $X_{1:m:n}^{(R_1,\ldots,R_m)}$ and the event $\{X_{1:m:n}^{(R_1,\ldots,R_m)} = X_{i:m:n}^{(R_1,\ldots,R_m)}\}$ are independent for some $2 \le i \le m$ iff F is a geometric cdf, that is, $F(j) = 1 - (1-p)^{j+1}$, $j = 0, 1, \ldots$, for some $p \in (0, 1)$;*
(2) *If S has at least three elements and for some $2 \le i < j \le m$, the rv $X_{i:m:n}^{(R_1,\ldots,R_m)}$ and the event $\{X_{j:m:n}^{(R_1,\ldots,R_m)} = X_{i:m:n}^{(R_1,\ldots,R_m)}\}$ are conditionally independent, given the event $\{X_{i:m:n}^{(R_1,\ldots,R_m)} > X_{i-1:m:n}^{(R_1,\ldots,R_m)}\}$, then F is a modified geometric distribution with $S = \{a_i \in \mathbb{R} | i = 0, 1, 2, \ldots$ and $a_i < a_j$ for $i < j\}$, $F(a_0) = \theta$ and $F(a_j) = 1 - (1-\theta)(1-p)^j$, $j = 1, 2, \ldots$, for some $\theta \in (0, 1)$ and $p \in (0, 1)$;*
(3) *If $S = \{0, 1, \ldots\}$, then for some $1 \le i < m$*

$$E\left(X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)} | X_{i:m:n}^{(R_1,\ldots,R_m)}, X_{i+1:m:n}^{(R_1,\ldots,R_m)} > X_{i:m:n}^{(R_1,\ldots,R_m)}\right) \text{ is constant a.s.} \quad (1)$$

iff F is a modified geometric distribution with cdf $F(0) = \theta$ and $F(j) = 1 - (1-\theta)(1-p)^j$, $j = 1, 2, \ldots$, for some $\theta \in (0, 1)$ and $p \in (0, 1)$;
(4) *If $S = \{0, 1, \ldots\}$ and $P(X_1 = 0) \ge (\le) P(X_1 = x)/P(X_1 \ge x)$ for all $x \in S$, then for some $1 \le i < j \le m$, the conditional distribution of the rv $\{X_{j:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)}\}$, given the event $\{X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)} > 0\}$, is the same as the unconditional distribution of $X_{j-i:m-i:\tilde{n}}^{(R_{i+1},\ldots,R_m)} + 1$, where $\tilde{n} = n - R_1 - R_2 - \cdots - R_i - i$, iff F is a geometric cdf.*

In fact, Tran [7 *Corollary* 3.5.3] *claimed that the property given in Equation* (1) *characterizes the geometric distribution, but it is a characteristic of a wider class of modified geometric cdfs* (*Section* 3).

Since the model of progressively Type-II right-censored order statistics reduces to ordinary order statistics when $(R_1, R_2, \dots, R_m) = (0, 0, \dots, 0)$, all the characterizations given in Theorem 1 are extensions of the corresponding characterizations based on ordinary order statistics. In this paper, we show that some other characterizations of discrete distributions via properties of ordinary order statistics can be generalized to characterizations involving progressively Type-II right-censored order statistics. The present paper is organized as follows. In Section 2, we recall some distributional properties of progressively Type-II right-censored order statistics and cite Shanbhag's lemma. Then, we establish our main results in Sections 3–5, concerning characterizations through conditional expectation, identical distribution, and independence of functions of progressively Type-II right-censored order statistics, respectively. Finally, we sketch extensions of these results to generalized order statistics in Section 6. It needs to be mentioned here that, for non-continuous distributions, generalized order statistics do not cover important submodels like record values and weak record values [8,9]. Since the most important settings included for discrete distributions are order statistics and progressively Type-II right-censored order statistics, we present the derivations for the latter situation and comment only on changes in the proofs for generalized order statistics.

Throughout the paper, for simplicity in notation, whenever it is clear what the progressive Type-II censoring scheme is, we will simply write $X_{i:m:n}$ instead of $X_{i:m:n}^{(R_1,\dots,R_m)}$. Moreover, whenever we use a conditional expectation, we will assume that this expectation exists. For brevity, we write $F(x-) = P(X < x)$, $x \in S$.

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2. Preliminary results

The key result that enables the study of progressively Type-II right-censored order statistics arising from discrete F is the following quantile representation due to Balakrishnan and Dembińska [3] (see also [6,10]).

Representation 1 *Let* $X_{1:m:n}^{(R_1,\dots,R_m)} \le \dots \le X_{m:m:n}^{(R_1,\dots,R_m)}$ *be progressively Type-II right-censored order statistics from any cdf* F*, and* $U_{1:m:n}^{(R_1,\dots,R_m)} \le \dots \le U_{m:m:n}^{(R_1,\dots,R_m)}$ *denote progressively Type-II right-censored order statistics (with the same censoring scheme) from the uniform distribution on* $[0, 1]$. *Define the generalized inverse function* $F^{\leftarrow}(\cdot)$ *by*

$$F^{\leftarrow}(u) = \inf\{x : F(x) \ge u\}, \quad u \in (0, 1).$$

Then,

$$\left(X_{1:m:n}^{(R_1,\dots,R_m)}, \dots, X_{m:m:n}^{(R_1,\dots,R_m)}\right) \stackrel{d}{=} \left(F^{\leftarrow}\left(U_{1:m:n}^{(R_1,\dots,R_m)}\right), \dots, F^{\leftarrow}\left(U_{m:m:n}^{(R_1,\dots,R_m)}\right)\right),$$

where $\stackrel{d}{=}$ *stands for equality in distribution.*

To exploit the above representation, we need expressions for the marginal and joint probability density functions (pdfs) of progressively Type-II right-censored order statistics from the uniform distribution on [0, 1]. These pdfs are given by

$$f_{U_{i:m:n}}(u) = c_i \sum_{j=1}^{i} a_{j,i}(1-u)^{\gamma_j - 1}, \quad 0 < u < 1, \ \ 1 \le i \le m, \tag{2}$$

$$f_{U_{i:m:n},U_{j:m:n}}(u_i, u_j) = c_j \left[\sum_{s=i+1}^{j} \left(\prod_{l=i+1, l\neq s}^{j} \frac{1}{\gamma_l - \gamma_s} \right) \left(\frac{1-u_j}{1-u_i} \right)^{\gamma_s} \right] \times \left[\sum_{s=1}^{i} a_{s,i}(1-u_i)^{\gamma_s} \right] \frac{1}{(1-u_i)(1-u_j)},$$
$$0 < u_i \le u_j < 1, \quad 1 \le i < j \le m, \tag{3}$$

where $\gamma_l = \sum_{j=l}^{m}(R_j + 1)$ and $c_l = \prod_{s=1}^{l} \gamma_s$, $1 \le l \le m$, $a_{s,l} = \prod_{r=1, r\neq s}^{l} \frac{1}{\gamma_r - \gamma_s}$, $1 \le s \le l \le m$, with the empty product $\Pi_\emptyset$ defined to be 1 [3, pp. 8–9; 11, Lemma 1].

We will also need the following result known in the literature as Shanbhag's lemma. Its proof can be found in Rao and Shanbhag [12, p. 28].

THEOREM 2 (Shanbhag's lemma) *Let* $\{(v_n, w_n) : n = 0, 1, \ldots)\}$ *be a sequence of vectors with non-negative real components such that* $v_n \neq 0$ *at least for one* $n \ge 1$ *and* $w_1 \neq 0$. *Then,*

$$v_n = \sum_{m=0}^{\infty} w_m v_{m+n}, \quad n = 0, 1, \ldots$$

iff $\sum_{m=0}^{\infty} w_m b^m = 1$ *and* $v_n = v_0 b^n$, $n = 1, 2, \ldots$, *for some* $b > 0$.

3. Characterizations via regression

As mentioned in Section 1, Tran [7, Corollary 3.5.3] claimed that the constancy of regression in Equation (1) characterizes the geometric distribution in the class of distributions having support on non-negative integers. Here, a correct version of her result is established. Moreover, our version is more general: the conditioning event $\{X_{i+1:m:n}^{(R_1,\ldots,R_m)} > X_{i:m:n}^{(R_1,\ldots,R_m)}\}$ in Equation (1) is replaced by $\{X_{i+1:m:n}^{(R_1,\ldots,R_m)} > X_{i:m:n}^{(R_1,\ldots,R_m)} + l\}$ with some fixed $l \in \{0, 1, \ldots\}$. Furthermore, $\Phi(X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)})$ replaces $X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)}$, where $(\Phi(n))_n$ is a suitably chosen sequence.

THEOREM 3 *Let* $X_{1:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ *be progressively Type-II right-censored order statistics from cdf* F *with support* $\{0, 1, \ldots\}$. *Fix* $l \in \{0, 1, \ldots\}$ *and suppose* $(\Phi(n))_{n\ge l}$ *is a non-decreasing sequence such that* $\Phi(l+2) > \Phi(l+1)$. *Then, for some* $1 \le i < m$,

$$E\left[\Phi\left(X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)}\right) \mid X_{i:m:n}^{(R_1,\ldots,R_m)}, X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)} > l\right]$$
$$\text{is constant a.s.} \tag{4}$$

iff F *is a modified geometric distribution with* $F(l+j) = 1 - (1-\theta)q^j$, $j \ge 0$, *for some* $\theta \in (0, 1)$ *and* $q \in (0, 1)$.

Proof Representation 1 and Equation (3) imply that

$$\begin{aligned} P(X_{i:m:n} = j, X_{i+1:m:n} > j + l) &= P(F^{\leftarrow}(U_{i:m:n}) = j, F^{\leftarrow}(U_{i+1:m:n}) > j + l) \\ &= P(F(j^-) < U_{i:m:n} \le F(j), U_{i+1:m:n} > F(j+l)) \\ &= c_{i+1} \int_{F(j^-)}^{F(j)} \sum_{s=1}^{i} a_{s,i}(1-u_i)^{\gamma_s - \gamma_{i+1} - 1}\, \mathrm{d}u_i \int_{F(j+l)}^{1} (1-u_{i+1})^{\gamma_{i+1}-1}\, \mathrm{d}u_{i+1}. \end{aligned} \tag{5}$$

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Similarly, we get for any $y \in \{1, 2, \ldots\}$

$$P(X_{i:m:n} = j, X_{i+1:m:n} = j + y)$$
$$= c_{i+1} \int_{F(j^-)}^{F(j)} \sum_{s=1}^{i} a_{s,i} (1 - u_i)^{\gamma_s - \gamma_{i+1} - 1} \, \mathrm{d}u_i \int_{F(j+y^-)}^{F(j+y)} (1 - u_{i+1})^{\gamma_{i+1} - 1} \, \mathrm{d}u_{i+1}. \tag{6}$$

It then follows that

$$P(X_{i+1:m:n} - X_{i:m:n} = y | X_{i:m:n} = j, X_{i+1:m:n} - X_{i:m:n} > l)$$
$$= \frac{P(X_{i:m:n} = j, X_{i+1:m:n} = j + y)}{P(X_{i:m:n} = j, X_{i+1:m:n} > j + l)} = \frac{\overline{F}^{\gamma_{i+1}}(j + y - 1) - \overline{F}^{\gamma_{i+1}}(j + y)}{\overline{F}^{\gamma_{i+1}}(j + l)}, \quad y \geq l + 1, \tag{7}$$

where $\overline{F} = 1 - F$. Thence, Equation (4) is equivalent to

$$\sum_{y=l+1}^{\infty} \Phi(y) \left[\overline{F}^{\gamma_{i+1}}(j + y - 1) - \overline{F}^{\gamma_{i+1}}(j + y) \right] = c \overline{F}^{\gamma_{i+1}}(j + l), \quad j \geq 0, \tag{8}$$

where c is some real constant. After some rearrangement, Equation (8) can be rewritten as

$$\sum_{y=0}^{\infty} [\Phi(y + l + 1) - \Phi(y + l)] \, \overline{F}^{\gamma_{i+1}}(j + y + l) = [c - \Phi(l)] \overline{F}^{\gamma_{i+1}}(j + l), \quad j \geq 0. \tag{9}$$

Since the sequence $(\Phi(n))_{n \geq l}$ is non-decreasing and $\Phi(l + 2) > \Phi(l + 1)$, Equation (9) shows that $c - \Phi(l) > 0$. Dividing Equation (9) by $c - \Phi(l)$ and substituting $v_j = \overline{F}^{\gamma_{i+1}}(j + l)$ and $w_y = [\Phi(y + l + 1) - \Phi(y + l)]/[c - \Phi(l)]$, we obtain

$$v_j = \sum_{y=0}^{\infty} w_y v_{y+j}, \quad j \geq 0.$$

Thus, conditions of Shanbhag's lemma hold and so by applying that result, we conclude that the constant regression property in Equation (4) is equivalent to

$$\overline{F}(j + l) = \overline{F}(l) q^j, \quad j = 1, 2, \ldots,$$

which reveals that F is a modified geometric distribution. ■

Remark 1 It is of interest to mention that, for ordinary order statistics, Theorem 3 has been proved by Rao and Shanbhag [13] for $l = 0$ and by Nagaraja [14] for $\Phi(j) - j \equiv 0$. However, with $l > 0$ and $\Phi(j) - j \not\equiv 0$, the established result is new even in the set-up of order statistics.

If we put $l < 0$ in Equation (4), then $\{X_{i+1:m:n} - X_{i:m:n} > l\}$ becomes a sure event, ties among $X_{i:m:n}$ and $X_{i+1:m:n}$ are permitted, and consequently results for $l < 0$ differ substantially from those for $l \geq 0$. For ordinary order statistics, Nagaraja [14] showed that in the geometric case

$$E(X_{i+1:n} - X_{i:n} | X_{i:n}) = \text{constant a.s.} \tag{10}$$

iff $i = 1$. López-Blázquez and Salamanca Miño [15] proved that if Equation (10) holds with $i = 1$ and the underlying cdf has support on non-negative integers, then F is geometric. The following theorem is a generalization of the latter result to the case of progressively Type-II right-censored order statistics.

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THEOREM 4 *Let* $X_{1:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ *be progressively Type-II right-censored order statistics from cdf F with support* $\{0, 1, \ldots, N\}$ *for some* $1 \le N \le \infty$. *Then,*

$$E\left(X_{2:m:n}^{(R_1,\ldots,R_m)} - X_{1:m:n}^{(R_1,\ldots,R_m)} | X_{1:m:n}^{(R_1,\ldots,R_m)}\right) = \text{constant a.s.} \tag{11}$$

iff F is geometric, that is, $F(j) = 1 - q^{j+1}$, $j \ge 0$, *for some* $q \in (0, 1)$.

Proof Since $X_{1:m:n} = X_{1:n}$, we immediately obtain

$$P(X_{1:m:n} = j) = P(X_{1:n} \ge j) - P(X_{1:n} \ge j+1) = \overline{F}^n(j-1) - \overline{F}^n(j), \quad j \ge 0. \tag{12}$$

By Equation (6), we have for $j \ge 0$ and $i \ge 1$

$$\begin{aligned} &P(X_{1:m:n} = j, X_{2:m:n} = j+i) \\ &= c_2 \int_{F(j^-)}^{F(j)} (1-u_1)^{R_1}\, du_1 \int_{F(j+i^-)}^{F(j+i)} (1-u_2)^{n-R_1-2}\, du_2 \\ &= \frac{n}{R_1+1}\left[\overline{F}^{R_1+1}(j-1) - \overline{F}^{R_1+1}(j)\right]\left[\overline{F}^{n-R_1-1}(j+i-1) - \overline{F}^{n-R_1-1}(j+i)\right]. \end{aligned} \tag{13}$$

Hence, for $j = 0, 1, \ldots, N$,

$$\begin{aligned} &E(X_{2:m:n} - X_{1:m:n} | X_{1:m:n} = j) \\ &= \sum_{i=1}^{\infty} i\, \frac{P(X_{1:m:n} = j, X_{2:m:n} = j+i)}{P(X_{1:m:n} = j)} \\ &= \frac{n}{R_1+1} \frac{\overline{F}^{R_1+1}(j-1) - \overline{F}^{R_1+1}(j)}{\overline{F}^n(j-1) - \overline{F}^n(j)} \sum_{i=1}^{\infty} i\left[\overline{F}^{n-R_1-1}(j+i-1) - \overline{F}^{n-R_1-1}(j+i)\right]. \end{aligned} \tag{14}$$

Since the sum on the right-hand side of Equation (14) is equal to $\sum_{i=1}^{\infty} \overline{F}^{n-R_1-1}(j+i-1)$, the constant regression property in Equation (11) yields

$$\frac{n}{R_1+1} \sum_{i=1}^{\infty} \overline{F}^{n-R_1-1}(j+i-1) = c\frac{\overline{F}^n(j-1) - \overline{F}^n(j)}{\overline{F}^{R_1+1}(j-1) - \overline{F}^{R_1+1}(j)}, \quad j = 0, 1, \ldots, N, \tag{15}$$

where c is some positive constant. Now observe that $N = \infty$, since otherwise for $j = N < \infty$ the right-hand side of Equation (15) is positive, while the left-hand side equals 0 and we have a contradiction.

Rewriting Equation (15) for $j-1$ instead of j and then taking the difference, we obtain for $j \ge 1$

$$\frac{n}{c(R_1+1)}\overline{F}^{n-R_1-1}(j-1) = \frac{\overline{F}^n(j-2) - \overline{F}^n(j-1)}{\overline{F}^{R_1+1}(j-2) - \overline{F}^{R_1+1}(j-1)} - \frac{\overline{F}^n(j-1) - \overline{F}^n(j)}{\overline{F}^{R_1+1}(j-1) - \overline{F}^{R_1+1}(j)}. \tag{16}$$

Dividing both sides of Equation (16) by $\overline{F}^{n-R_1-1}(j-1)$ and denoting $\overline{F}(j)/\overline{F}(j-1)$ by v_j, we get

$$\frac{n}{c(R_1+1)} = \frac{v_{j-1}^{-n} - 1}{v_{j-1}^{-R_1-1} - 1} - \frac{1 - v_j^n}{1 - v_j^{R_1+1}}, \quad j \ge 1, \tag{17}$$

where $v_j \in (0, 1)$, $j \ge 0$. To solve the above difference equation, we apply a method similar to that used by López-Blázquez and Salamanca Miño [15]. Let $P_n(x) = (1 - x^n)/(1 - x^{R_1+1})$ and

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$Q_n(x) = P_n(1/x)$, $x \in (0, 1)$. Then, Equation (17) takes on the form

$$\frac{n}{c(R_1+1)} = Q_n(v_{j-1}) - P_n(v_j), \quad j \geq 1. \tag{18}$$

Using the geometric sum representation, we get

$$P_n(x) = \frac{\sum_{j=0}^{n-1} x^j}{\sum_{j=0}^{R_1} x^j} = 1 + \frac{\sum_{j=R_1+1}^{n-1} x^j}{\sum_{j=0}^{R_1} x^j} = 1 + \frac{\sum_{j=1}^{n-R_1-1} x^j}{\sum_{j=0}^{R_1} x^{-j}}, \quad x \in (0, 1). \tag{19}$$

Thus, P_n is a strictly increasing function on (0, 1) so that $P_n'(x) > 0$, $x \in (0, 1)$. Consequently,

$$[Q_n(x) - P_n(x)]' = -\left[P_n'\left(\frac{1}{x}\right)x^{-2} + P_n'(x)\right] < 0, \quad x \in (0, 1),$$

which means that $Q_n - P_n$ is strictly decreasing on (0, 1). Moreover, $Q_n(1) - P_n(1) = 0$ and $\lim_{x\to 0^+}[Q_n(x) - P_n(x)] = \lim_{x\to\infty} P_n(x) - 1 = \infty$. Hence, there exists exactly one $q \in (0, 1)$ satisfying

$$\frac{n}{c(R_1+1)} = Q_n(q) - P_n(q). \tag{20}$$

Now, we show that the only solution of the difference Equation (18) is given by $v_j = q$, $j \geq 0$. To do this, note that P_n' is strictly increasing in (0, 1), i.e. P_n is a convex function. For $R_1 = 0$, P_n is a convex polynomial on [0, 1] (see (Equation 19)). Thus, we can assume $R_1 > 0$. Then,

$$\begin{aligned} P_n'(x) &= \frac{\sum_{j=1}^{n-R_1-1} jx^{R_1+j-1}}{\sum_{j=0}^{R_1} x^j} + \frac{\left(\sum_{j=0}^{R_1-1}(R_1-j)x^j\right)\left(\sum_{j=R_1}^{n-2} x^j\right)}{\left(\sum_{j=0}^{R_1} x^j\right)^2} \\ &= \frac{\sum_{j=1}^{n-R_1-1} jx^{j-1}}{\sum_{j=0}^{R_1} x^{-j}} + \frac{\left(\sum_{j=0}^{n-R_1-2} x^j\right)\left(\sum_{j=0}^{R_1-1}(R_1-j)x^j\right)}{\left(\sum_{j=0}^{R_1} x^j\right)\left(\sum_{j=0}^{R_1} x^{-j}\right)}. \end{aligned}$$

The first component of the above sum and the numerator of the second component are strictly increasing. In order to prove that P_n' is strictly increasing, it suffices to show that the denominator

$$D(x) = \left(\sum_{j=0}^{R_1} x^j\right)\left(\sum_{j=0}^{R_1} x^{-j}\right) = \frac{1}{x^{R_1}}\left(\sum_{j=0}^{R_1} x^j\right)^2, \quad x \in (0, 1),$$

is decreasing in x. In fact, taking into account the identity

$$\left(\sum_{j=0}^{R_1} x^j\right)^2 = \sum_{j=0}^{R_1-1}(j+1)(x^j + x^{2R_1-j}) + (R_1+1)x^{R_1},$$

we arrive at the expression

$$D(x) = R_1 + 1 + \sum_{j=0}^{R_1-1}(j+1)(x^{j-R_1} + x^{R_1-j}), \quad x \in (0, 1),$$

which is easily seen to be strictly decreasing on the interval (0, 1).

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This completes the proof of the monotonicity of P_n'. Moreover, this implies that

$$P_n'(x) < P_n'(1) \quad \text{for all } x \in (0,1). \tag{21}$$

Furthermore, as a product of two positive strictly decreasing functions, $-Q_n'(x) = P_n'(1/x)x^{-2}$ is strictly decreasing on $(0, 1)$. Consequently,

$$-Q_n'(x) > -Q_n'(1) \quad \text{for all } x \in (0,1). \tag{22}$$

Since $P_n'(1) = Q_n'(1)$, the inequalities in Equations (21) and (22) show that

$$\frac{-Q_n'(y)}{P_n'(x)} > \frac{-Q_n'(1)}{P_n'(1)} = 1, \text{ which readily implies } \quad -Q_n'(y) > P_n'(x) \text{ for all } x, y \in (0,1). \tag{23}$$

Now, upon subtracting Equation (18) from (20), we obtain

$$Q_n(v_{j-1}) - Q_n(q) = P_n(v_j) - P_n(q), \quad j \geq 1. \tag{24}$$

For $v_0 = q$, the monotonicity of P_n and Equation (24) yield that $v_j = q$ for all $j \geq 1$, which implies $F(j) = 1 - q^{j+1}$, $j \geq 0$. On the other hand, for $v_0 < q$, from Equations (23) and (24) and repeating the method used in the proof of Lemma 2 of López-Blázquez and Salamanca Miño [15], we get a contradiction. In the same manner, a contradiction results for $v_0 > q$. Thus, the 'only if' part of the theorem is established. To show the 'if' part, we substitute $\overline{F}(j) = q^{j+1}$, $j \geq 0$ into Equation (14), which shows that the regression in question is constant in j, and indeed

$$E(X_{2:m:n} - X_{1:m:n} | X_{1:m:n} = j) = \frac{n}{R_1 + 1} \frac{q^{n-R_1-1}(1 - q^{R_1+1})}{(1 - q^n)(1 - q^{n-R_1-1})}. \qquad \blacksquare$$

4. A characterization via identical distribution

Generalizing an earlier work of Puri and Rubin [16], Zijlstra [17] showed that, if the support of X_1 is a subset of non-negative integers and if $P(X_1 = 1) > 0$ and $P(X_1 > 1) > 0$, then the rvs $X_{k+1:n} - X_{k:n}$ and $X_{1:n-k}$ have the same distribution iff F is a modified geometric distribution (see also [18]). The following theorem establishes a more general result in the set-up of progressively Type-II right-censored order statistics.

THEOREM 5 *Let* $X_{1:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ *be progressively Type-II right-censored order statistics from cdf F with support being a subset of non-negative integers. Assume* $P(X_1 = 1) > 0$ *and* $P(X_1 > 1) > 0$*. Then, for some* $1 \leq i < m$,

$$X_{i+1:m:n}^{(R_1,\ldots,R_m)} - X_{i:m:n}^{(R_1,\ldots,R_m)} \stackrel{d}{=} X_{1:m-i:\gamma_{i+1}}^{(R_{i+1},\ldots,R_m)} \tag{25}$$

iff F is a modified geometric distribution with cdf $F(j) = 1 - (1 - p_0)q^j$, $j \geq 0$, *where* $p_0 \in (0, 1)$ *and* $q \in (0, 1)$ *are such that*

$$1 = c_i \sum_{s=1}^{i} \left[\frac{a_{s,i}}{\gamma_s - \gamma_{i+1}} \left(1 - (1 - p_0)^{\gamma_s - \gamma_{i+1}} \frac{1 - q^{\gamma_{i+1}}}{1 - q^{\gamma_s}} \right) \right]. \tag{26}$$

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Proof From Equation (5), we obtain

$$\begin{aligned} &P\left(X_{i+1:m:n}^{(R_1,\dots,R_m)} - X_{i:m:n}^{(R_1,\dots,R_m)} > k\right) \\ &= \sum_{j=0}^{\infty} P\left(X_{i:m:n}^{(R_1,\dots,R_m)} = j,\, X_{i+1:m:n}^{(R_1,\dots,R_m)} > j+k\right) \\ &= \frac{c_{i+1}}{\gamma_{i+1}} \sum_{j=0}^{\infty}\left[\overline{F}^{\gamma_{i+1}}(j+k)\sum_{s=1}^{i}\left(a_{s,i}\int_{F(j^-)}^{F(j)}(1-u_i)^{\gamma_s-\gamma_{i+1}-1}\,\mathrm{d}u_i\right)\right], \quad k \ge 0. \end{aligned}$$

Denoting now $\overline{F}^{\gamma_{i+1}}(k)$ by v_k and $(c_{i+1}/\gamma_{i+1})\sum_{s=1}^{i}\left(a_{s,i}\int_{F(j^-)}^{F(j)}(1-u_i)^{\gamma_s-\gamma_{i+1}-1}\,\mathrm{d}u_i\right)$ by w_j, we can express Equation (25) as

$$v_k = \sum_{j=0}^{\infty} w_j v_{k+j}, \quad k \ge 0, \tag{27}$$

since $P(X_{1:m-i:\gamma_{i+1}}^{(R_{i+1},\dots,R_m)} > k) = P(X_{1:\gamma_{i+1}} > k) = \overline{F}^{\gamma_{i+1}}(k)$, $k \ge 0$. Notice that the conditions of Shanbhag's lemma are satisfied. Indeed,

$$\begin{aligned} v_1 &= [P(X_1 > 1)]^{\gamma_{i+1}} > 0, \\ w_j &= \int_{F(j^-)}^{F(j)} c_i \sum_{s=1}^{i} a_{s,i}(1-u_i)^{\gamma_s-1}(1-u_i)^{-\gamma_{i+1}}\,\mathrm{d}u_i \\ &= \int_{F(j^-)}^{F(j)} f_{U_{i:m:n}^{(R_1,\dots,R_m)}}(u_i)(1-u_i)^{-\gamma_{i+1}}\,\mathrm{d}u_i \ge 0, \end{aligned}$$

and

$$w_1 = \int_{F(0)}^{F(1)} f_{U_{i:m:n}^{(R_1,\dots,R_m)}}(u_i)(1-u_i)^{-\gamma_{i+1}}\,\mathrm{d}u_i > 0,$$

because the density of $U_{i:m:n}^{(R_1,\dots,R_m)}$ is positive on $(0, 1)$ and by assumption, $F(1) - F(0) = P(X_1 = 1) > 0$.

On applying Shanbhag's lemma, we conclude that Equation (27) holds iff

$$\overline{F}(k) = \overline{F}(0)q^k, \ \ k \ge 1 \quad \text{and} \quad \sum_{j=0}^{\infty} w_j q^{j\gamma_{i+1}} = 1,$$

which implies that

$$F(k) = 1 - (1-p_0)q^k, \quad k \ge 0, \tag{28}$$

where $p_0 = P(X_1 = 0)$ and

$$\frac{1}{c_i} = \sum_{s=1}^{i} \frac{a_{s,i}}{\gamma_s - \gamma_{i+1}}\left[\sum_{j=0}^{\infty} q^{j\gamma_{i+1}}\left(\overline{F}^{\gamma_s-\gamma_{i+1}}(j-1) - \overline{F}^{\gamma_s-\gamma_{i+1}}(j)\right)\right]. \tag{29}$$

Substituting the expression in Equation (28) into Equation (29), we obtain Equation (26).

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It still remains to be shown that for any n, m, $i \in \{1, \ldots, m-1\}$ and censoring scheme $(R_1, \ldots, R_m)$ such that $n = R_1 + \ldots + R_m + m$, there exist $p_0 \in (0, 1)$ and $q \in (0, 1)$ satisfying Equation (26). For this purpose, we introduce $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = c_i \sum_{s=1}^{i} \left\{ \frac{a_{s,i}}{\gamma_s - \gamma_{i+1}} \left[1 - (1-t)^{\gamma_s - \gamma_{i+1}}\right] \right\}, \quad t \in [0, 1].$$

We then have

$$g(1) = \prod_{l=1}^{i} \frac{\gamma_l}{\gamma_l - \gamma_{i+1}} > 1. \tag{30}$$

This follows directly from the following argument: let $\eta_s = \gamma_s - \gamma_{i+1}$, $1 \le s \le i$. Then, $a_{s,i} = \prod_{l=1, l \neq s}^{i} \frac{1}{\eta_l - \eta_s} = \tilde{a}_{s,i}$ so that

$$g(1) = c_i \sum_{s=1}^{i} \frac{\tilde{a}_{s,i}}{\eta_s} = \frac{c_i}{\prod_{s=1}^{i} \eta_s} \prod_{s=1}^{i} \eta_s \sum_{s=1}^{i} \frac{\tilde{a}_{s,i}}{\eta_s} = \frac{c_i}{\prod_{s=1}^{i} \eta_s}(1 - F_{i*}(0)) = \frac{c_i}{\prod_{s=1}^{i} \eta_s},$$

where F_{i*} denotes the cdf of the ith uniform progressively Type-II right-censored order statistic based on $(R_1, \ldots, R_i)$ in a sample of size $\gamma_1 - \gamma_{i+1} = n - \gamma_{i+1}$ (see also [11]). Since $g(0) = 0$ and g is a continuous function, Equation (30) implies the existence of $p_0 \in (0, 1)$ with $g(p_0) = 1$. The proof is completed by just observing that the right-hand side of Equation (26) tends to $g(p_0)$ for $q \to 0$. ■

5. Characterizations based on independence

It is well known that in the geometric case the rvs $X_{1:n}$ and $X_{i:n} - X_{1:n}$ are independent for any $1 < i \le n$. Several characterizations based on this property and its weaker versions are available in the literature (see [19, Section 4.2] for a recent account of such results). In this section, we consider two extensions of these results to the case of progressively Type-II right-censored order statistics. The first one is a generalization of a theorem due to Govindarajulu [20, Theorem 1].

THEOREM 6 *Let $X_{1:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ be progressively Type-II right-censored order statistics from cdf F with support $\{0, 1, \ldots\}$. Then, for a fixed $k \ge 2$ and $1 < i \le m$, the rv $X_{1:m:n}^{(R_1,\ldots,R_m)}$ and the event $\{X_{i:m:n}^{(R_1,\ldots,R_m)} - X_{1:m:n}^{(R_1,\ldots,R_m)} \ge k\}$ are independent and*

$$\overline{F}(j) = q^{j+1}, \quad 0 \le j < k, \tag{31}$$

iff F is a geometric distribution.

Proof Representation 1 and Equation (3) yield

$$\begin{aligned} &P(X_{1:m:n} = j, X_{i:m:n} - X_{1:m:n} \ge k) \\ &\quad = P(F^{\leftarrow}(U_{1:m:n}) = j, F^{\leftarrow}(U_{i:m:n}) \ge k + j) \\ &\quad = P(F(j^-) < U_{1:m:n} \le F(j), U_{i:m:n} > F(k + j - 1)) \end{aligned}$$

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$$= c_i \sum_{s=2}^{i} \left[\left(\prod_{l=2,l\neq s}^{i} \frac{1}{\gamma_l - \gamma_s} \right) \int_{F(k+j-1)}^{1} (1-u_i)^{\gamma_s - 1}\, \mathrm{d}u_i \int_{F(j^-)}^{F(j)} (1-u_1)^{\gamma_1 - \gamma_s - 1}\, \mathrm{d}u_1 \right]$$

$$= c_i \int_{F(j^-)}^{F(j)} (1-u_1)^{n-1} \sum_{s=2}^{i} \left[\left(\prod_{l=2,l\neq s}^{i} \frac{1}{\gamma_l - \gamma_s} \right) \frac{1}{\gamma_s} \left(\frac{\overline{F}(k+j-1)}{1-u_1} \right)^{\gamma_s} \right] \mathrm{d}u_1$$

$$= c_i \overline{F}^n (k+j-1) \int_{\overline{F}(k+j-1)/\overline{F}(j-1)}^{\overline{F}(k+j-1)/\overline{F}(j)} t^{-n-1} G_i(t)\, \mathrm{d}t,$$

where the last equality was obtained by substituting $1 - u_1 = t^{-1}\overline{F}(k+j-1)$, and G_i defined by

$$G_i(t) = \sum_{s=2}^{i} \left(\prod_{l=2,l\neq s}^{i} \frac{1}{\gamma_l - \gamma_s} \right) \frac{1}{\gamma_s} t^{\gamma_s} = \frac{n}{c_i} P \left(U_{i-1:m-1:n-R_1-1}^{(R_2,R_3,\ldots,R_m)} > 1-t \right), \quad t \in [0,1] \tag{32}$$

is a strictly increasing function on (0, 1).

Hence, by Equation (12), we get

$$P(X_{i:m:n} - X_{1:m:n} \geq k | X_{1:m:n} = j)$$
$$= c_i \left[\frac{\overline{F}(k+j-1)}{\overline{F}(j)} \right]^n \int_{\overline{F}(k+j-1)/\overline{F}(j-1)}^{\overline{F}(k+j-1)/\overline{F}(j)} t^{-n-1} G_i(t)\, \mathrm{d}t \Bigg/ \left[\frac{\overline{F}^n (j-1)}{\overline{F}^n (j)} - 1 \right], \quad j \geq 0. \tag{33}$$

The independence assumption implies that the expression in Equation (33) has the same value for $j = 0$ and $j = 1$. Therefore, by Equation (31), we have

$$q^n \int_{q^k}^{q^{k-1}} t^{-n-1} G_i(t)\, \mathrm{d}t = q_k^n \int_{q^{k-1} q_k}^{q^{k-2} q_k} t^{-n-1} G_i(t)\, \mathrm{d}t,$$

where $q_k = \overline{F}(k)/\overline{F}(k-1)$. Substituting $z = t/q$ and $z = t/q_k$, respectively, we arrive at the equation

$$\int_{q^{k-1}}^{q^{k-2}} z^{-n-1} G_i(qz)\, \mathrm{d}z = \int_{q^{k-1}}^{q^{k-2}} z^{-n-1} G_i(q_k z)\, \mathrm{d}z. \tag{34}$$

Now, let us assume that $q_k < q$. Since G_i is a strictly increasing function, the right-hand side of Equation (34) has to be necessarily less than its left-hand side, which is a contradiction. Given that $q_k > q$, we also get a contradiction by using a similar argument. Thus, $q_k = q$ in which case by recurrence we obtain $q_j = \overline{F}(j)/\overline{F}(j-1) = q$, $j \geq k$, which means that $\overline{F}(j) = q^{j+1}$, $j \geq 0$. This proves the converse of the assertion of the theorem. On the other hand, for the geometric distribution, $\overline{F}(j+i)/\overline{F}(j)$ does not depend on j, and consequently $\{X_{i:m:n} - X_{1:m:n} \geq k\}$ and $X_{1:m:n}$ are independent by Equation (33). ■

The above theorem assumes $k \geq 2$. If we consider $k = 1$, then, in the class of distributions supported on non-negative integers, we can replace the independence of the rv $X_{1:m:n}$ and the event $\{X_{i:m:n} - X_{1:m:n} \geq 1\}$ by its equivalent condition that $X_{1:m:n}$ and $\{X_{i:m:n} = X_{1:m:n}\}$ are independent. If we drop the assumption on the support of F, we obtain a characterization of the geometric-type distribution.

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THEOREM 7 *Let $X_{1:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ be progressively Type-II right-censored order statistics from cdf F with support $\{a_0, a_1, \ldots\}$, where $(a_n)_{n\geq 0}$ is an increasing sequence of real numbers. Then, the rv $X_{1:m:n}^{(R_1,\ldots,R_m)}$ and the event $\{X_{i:m:n}^{(R_1,\ldots,R_m)} = X_{1:m:n}^{(R_1,\ldots,R_m)}\}$ are independent for some $1 < i \leq m$ iff F is a geometric-type distribution, that is, $\overline{F}(a_j) = q^{j+1}$, $j \geq 0$, with $q \in (0, 1)$.*

Proof We first derive an expression for $P(X_{i:m:n} > X_{1:m:n}|X_{1:m:n} = a_j)$ because the assumed independence condition holds iff this conditional probability is free of j for all $j \geq 0$.

Representation 1, Equation (3) and arguments similar to those used in the proof of Theorem 6 lead to the expression

$$P(X_{i:m:n} > X_{1:m:n}|X_{1:m:n} = a_j) = c_i \overline{F}^n(a_j) \int_{\overline{F}(a_j)/\overline{F}(a_{j-1})}^{1} t^{-n-1} G_i(t)\,\mathrm{d}t, \quad j \geq 0,$$

where $a_{-1} = -\infty$ and $G_i(t)$ is as given in Equation (32). Moreover,

$$P(X_{1:m:n} = a_j) = \overline{F}^n(a_{j-1}) - \overline{F}^n(a_j) = n\overline{F}^n(a_j) \int_{\overline{F}(a_j)/\overline{F}(a_{j-1})}^{1} t^{-n-1}\,\mathrm{d}t, \quad j \geq 0.$$

Therefore, the required conditional probability is obtained as

$$P(X_{i:m:n} > X_{1:m:n}|X_{1:m:n} = a_j) = \frac{c_i}{n} H\left(\frac{\overline{F}(a_j)}{\overline{F}(a_{j-1})}\right), \tag{35}$$

where H is defined by

$$H(x) = \frac{\int_x^1 t^{-n-1} G_i(t)\,\mathrm{d}t}{\int_x^1 t^{-n-1}\,\mathrm{d}t}, \quad x \in (0, 1).$$

The function H is strictly increasing because, for $0 < x < 1$, the strict monotonicity of G_i yields

$$\begin{aligned} H'(x) &= \left(\int_x^1 t^{-n-1}\,\mathrm{d}t\right)^{-2} x^{-n-1} \left[\int_x^1 t^{-n-1} G_i(t)\,\mathrm{d}t - G_i(x) \int_x^1 t^{-n-1}\,\mathrm{d}t\right] \\ &= \left(\int_x^1 t^{-n-1}\,\mathrm{d}t\right)^{-2} x^{-n-1} \int_x^1 t^{-n-1} \underbrace{(G_i(t) - G_i(x))}_{>0 \text{ for } t>x}\,\mathrm{d}t > 0, \quad x \in (0, 1). \end{aligned}$$

The strict monotonicity of H implies that the expression in Equation (35) does not depend on j iff the ratio $\overline{F}(a_j)/\overline{F}(a_{j-1})$ is constant for all $j \geq 0$, that is, iff F is a geometric-type distribution. ∎

In the case of ordinary order statistics, Theorem 7 was proved by El-Neweihi and Govindarajulu [21, Theorem 3.1]. Tran [7, Theorem 3.3.1] extended their result to generalized order statistics (and hence to progressively Type-II right-censored order statistics), but she made a stronger assumption that the support of the underlying distribution consists of all non-negative integers.

6. Extensions to generalized order statistics

In this section, we illustrate how the present results for progressively Type-II right-censored order statistics can be extended to generalized order statistics. If the result reads the same, we comment only on differences in the proof.

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Progressively Type-II right-censored order statistics can be seen as particular generalized order statistics $X_*^{(1)}, \dots, X_*^{(m)}$ introduced by Kamps [4,5]. Here, the parameters $\gamma_1, \dots, \gamma_m$ need not be integers and, more importantly, they need not to be strictly ordered. Although the joint density function of uniform generalized order statistics (here given in terms of $\gamma_1, \dots, \gamma_m$ with $\gamma_{m+1} = 0$)

$$f^{U_*^{(1)}, \dots, U_*^{(m)}}(u_1, \dots, u_m) = \prod_{j=1}^{m} \gamma_j \prod_{j=1}^{m} (1 - u_j)^{\gamma_j - \gamma_{j+1} - 1}, \quad 0 \le u_1 \le \dots \le u_m < 1, \tag{36}$$

looks quite similar to that of progressively Type-II right-censored order statistics, the marginal distributions become complicated if some γ_js are equal. In this case, the one dimensional marginals can be expressed in terms of so-called Meijer's G-functions (see also [22]):

$$f^{U_i^{(*)}}(u) = \prod_{j=1}^{i} \gamma_j G_{i,i}^{i,0}(1 - u|\gamma_1, \dots, \gamma_i), \quad u \in (0, 1).$$

Moreover, according to Cramer [23], the bivariate marginal density function of two uniform generalized order statistics can be written in a similar manner. For $(U_i^{(*)}, U_{i+1}^{(*)})$, we get

$$f^{U_i^{(*)}, U_{i+1}^{(*)}}(u_i, u_{i+1}) = \gamma_{i+1}(1 - u_{i+1})^{\gamma_{i+1} - 1}(1 - u_i)^{-\gamma_i - 1} f^{U_i^{(*)}}(u_i) = (1 - u_{i+1})^{\gamma_{i+1} - 1} \xi(u_i).$$

Having this in mind, it is easy to see that Theorem 3 holds also for generalized order statistics. In the proof, we just have to rewrite Equations (5) and (6). In particular, Equation (5) reads for generalized order statistics as

$$P(X_*^{(i)} = j, X_*^{(i+1)} > j + l) = \left[\int_{F(j^-)}^{F(j)} \xi(u_i) \, \mathrm{d}u_i \right] \left[\int_{F(j+l)}^{1} (1 - u_{i+1})^{\gamma_{i+1} - 1} \, \mathrm{d}u_{i+1} \right]. \tag{37}$$

Therefore, Equation (7) is identical and the remaining parts of the proof coincide.

The generalization of Theorem 4 cannot be obtained directly from the progressive censoring result. First, we have to consider two cases $\gamma_1 \neq \gamma_2$ and $\gamma_1 = \gamma_2$. In both cases, analogous difference equations can be established (see Equation (17)). But, the question whether there is a (unique) solution remains open. Clearly, the proof applied in the progressive censoring context cannot be extended since it heavily depends on the assumptions that $\gamma_1 > \gamma_2$ and that both parameters are integer-valued.

Theorem 5 can be easily written in terms of generalized order statistics by adopting the respective notation and using Equation (37) instead of Equation (5) in the first step of the proof. It reads as follows.

THEOREM 8 *Let $X_*^{(1)}, \dots, X_*^{(m)}$ be generalized order statistics based on cdf F and parameters $\gamma_1, \dots, \gamma_m$ with support being a subset of non-negative integers. Assume $P(X_1 = 1) > 0$ and $P(X_1 > 1) > 0$. Then, for some $1 \le i < m$,*

$$X_*^{(i+1)} - X_*^{(i)} \stackrel{d}{=} Y_*^{(1)},$$

where $Y_*^{(1)}$ is a first generalized order statistic based on F and parameter γ_{i+1} iff F is a modified geometric distribution with cdf $F(j) = 1 - (1 - p_0)q^j$, $j \ge 0$, where $p_0 \in (0, 1)$ and $q \in (0, 1)$ are such that

$$1 = \sum_{j=1}^{\infty} q^j \int_{1-(1-p_0)q^{j-1}}^{1-(1-p_0)q^j} (1 - t)^{-\gamma_i - 1} f^{U_i^{(*)}}(t) \, \mathrm{d}t + \int_0^{p_0} (1 - t)^{-\gamma_i - 1} f^{U_i^{(*)}}(t) \, \mathrm{d}t. \tag{38}$$

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It has to be mentioned that Equation (38) simplifies for $\gamma_i \neq \gamma_j, i \neq j$. In this case, we obtain a simple equation similar to Equation (26).

Finally, Theorems 6 and 7 also hold for generalized order statistics. In order to see this, we establish an appropriate representation for the probability $P(X_*^{(1)} = j, X_*^{(i)} - X_*^{(1)} \geq k)$. First, we get

$$P(X_*^{(1)} = j, X_*^{(i)} - X_*^{(1)} \geq k) = P(F(j^-) < U_*^{(1)} \leq F(j), U_*^{(i)} > F(k+j-1)).$$

From Cramer and Kamps [22], we conclude that

$$U_*^{(i)} | U_*^{(1)} = t \sim 1 - (1-t)(1 - Z_*^{(i-1)}), \quad t \in (0,1),$$

where $Z_*^{(i-1)}$ denotes a uniform generalized order statistic based on parameters $\gamma_2, \ldots, \gamma_i$. Therefore, we find after some calculations

$$\begin{aligned} P(X_*^{(1)} = j, X_*^{(i)} - X_*^{(1)} \geq k) &= \gamma_1 \int_{F(j^-)}^{F(j)} P\left(Z_*^{(i-1)} > 1 - \frac{\overline{F}(k+j-1)}{1-t}\right)(1-t)^{\gamma_1 - 1}\,\mathrm{d}t \\ &= \gamma_1 \overline{F}^{\gamma_1}(k+j-1) \int_{\overline{F}(k+j-1)/F(j^-)}^{\overline{F}(k+j-1)/F(j)} P(Z_*^{(i-1)} > 1 - v) v^{-\gamma_1 - 1}\,\mathrm{d}v. \end{aligned}$$

Introducing the notation $G_i(t) = P(Z_*^{(i-1)} > 1 - v)$, we get the following identity analogous to Equation (33)

$$\begin{aligned} &P(X_*^{(i)} - X_*^{(1)} \geq k | X_*^{(1)} = j) \\ &= \left[\frac{\overline{F}(k+j-1)}{\overline{F}(j)}\right]^{\gamma_1} \int_{\overline{F}(k+j-1)/\overline{F}(j-1)}^{\overline{F}(k+j-1)/\overline{F}(j)} t^{-\gamma_1 - 1} G_i(t)\,\mathrm{d}t \Bigg/ \left[\frac{\overline{F}^{\gamma_1}(j-1)}{\overline{F}^{\gamma_1}(j)} - 1\right], \quad j \geq 0. \end{aligned} \tag{39}$$

Obviously, G_i is a strictly increasing function so that we can proceed as in the proof of Theorem 6. The changes in the proof of Theorem 7 are evident from the above comments.

References

[1] N. Balakrishnan and R. Aggarwala, *Progressive Censoring: Theory, Methods, and Applications*, Birkhäuser, Boston, MA, 2000.

[2] N. Balakrishnan, *Progressive censoring methodology: An appraisal (with discussions)*, Test 16 (2007), pp. 211–296.

[3] N. Balakrishnan and A. Dembińska, *Progressively Type-II right censored order statistics from discrete distributions*, J. Statist. Plann. Inference 138 (2008), pp. 845–856.

[4] U. Kamps, *A concept of generalized order statistics*, J. Statist. Plann. Inference 48 (1995a), pp. 1–23.

[5] U. Kamps, *A Concept of Generalized Order Statistics*, Teubner, Stuttgart, Germany, 1995b.

[6] T. Fischer, N. Balakrishnan, and E. Cramer, *Mixture representation for order statistics from INID progressive censoring and its applications*, J. Multivariate Anal. 99 (2008), pp. 1999–2015.

[7] T.T.H. Tran, *Discrete Generalized Order Statistics*, Ph.D. thesis, University of Oldenburg, Oldenburg, Germany, 2006.

[8] N. Balakrishnan and A. Dembińska, *Ordered random variables from discontinuous distributions*, J. Iran. Statist. Soc. 6 (2007), pp. 31–46.

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[9] E. Cramer and T.T.H. Tran, *Generalized order statistics from arbitrary distributions and the Markov chain property*, J. Statist. Plann. Inference 139 (2009) pp. 4064–4071.
[10] N. Balakrishnan and A. Dembińska, *Erratum to 'Progressively Type-II right censored order statistics from discrete distributions'* [J. Statist. Plann. Inference 138 (2008) 845–856], J. Statist. Plann. Inference 139 (2009), pp. 1572–1574.
[11] U. Kamps and E. Cramer, *On distributions of generalized order statistics*, Statistics 35 (2001), pp. 269–280.
[12] C.R. Rao and D.N. Shanbhag, *Choquet-Deny Type Functional Equations with Applications to Stochastic Models*, John Wiley & Sons, Chichester, UK, 1994.
[13] C.R. Rao and D.N. Shanbhag, *Recent approaches to characterizations based on order statistics and record values*, in *Handbook of Statistics, Vol 16 – Order Statistics: Theory & Methods*, N. Balakrishnan and C.R. Rao, eds., North-Holland, Amsterdam, The Netherlands, 1998, pp. 231–256.
[14] H.N. Nagaraja, *Some characterizations of discrete distributions based on linear regressions of adjacent order statistics*, J. Statist. Plann. Inference 20 (1988), pp. 65–75.
[15] F. López-Blázquez and B. Salamanca Miño, *A characterization of the geometric distribution*, Statist. Pap. 39 (1998), pp. 231–236.
[16] P. Puri and H. Rubin, *A characterization based on the absolute difference of two i.i.d. random variables*, Ann. Math. Stat. 41 (1970), pp. 2113–2122.
[17] M. Zijlstra, *Characterizations of the geometric distribution by distributional properties*, J. Appl. Probab. 20 (1983), pp. 843–850.
[18] B.C. Arnold, *Two characterizations of the geometric distribution*, J. Appl. Probab. 17 (1980), pp. 570–573.
[19] A. Dembińska, *Discrete order statistics*, in *Encyclopedia of Statistical Sciences*, S. Kotz et al., eds., Online version, John Wiley & Sons, Hoboken, NJ, 2008.
[20] Z. Govindarajulu, *Characterization of the geometric distribution using properties of order statistics*, J. Statist. Plann. Inference 4 (1980), pp. 237–247.
[21] E. El-Neweihi and Z. Govindarajulu, *Characterizations of geometric distribution and discrete IFR (DFR) distributions using order statistics*, J. Statist. Plann. Inference 3 (1979), pp. 85–90.
[22] E. Cramer and U. Kamps, *Marginal distributions of sequential and generalized order statistics*, Metrika 58 (2003), pp. 293–310.
[23] E. Cramer, *Dependence structure of generalized order statistics*, Statistics 40 (2006), pp. 409–413.

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