PSEUDO-RIEMANNIAN METRIC WITH NEUTRAL SIGNATURE INDUCED BY SOLUTIONS TO EULER–LAGRANGE EQUATIONS FOR A FIELD OF COMPLEX LINEAR FRAMES

PIOTR GODLEWSKI

Faculty of Mathematics and Information Science,
Warsaw University of Technology,
Plac Politechniki 1, 00-661 Warszawa, Poland
(e-mail: pgod@mini.pw.edu.pl)

(*Received October 1, 2012 – Revised March 12, 2013*)

We investigate a model of the field of complex linear frames $\boldsymbol{E}$ on the product manifold $M = \mathbb{R} \times \mathrm{G}$, where G is a real semisimple Lie group. The model is invariant under the natural action of the group $\mathrm{GL}(n, \mathbb{C})$ $(n = \dim M)$. It results in a modified Born–Infeld-type nonlinearity of Euler–Lagrange equations.

We analyse a family of solutions to Euler–Lagrange equations. Each solution $\boldsymbol{E}$ belonging to this family induces a pseudo-Riemannian metric $\gamma[\boldsymbol{E}]$ on $M = \mathbb{R} \times \mathrm{G}$. In the physical case where $n = \dim M = 4$, among these solutions there exist ones for which the signature of $\gamma[\boldsymbol{E}]$ is neutral $(+ + - -)$.

The existence of solutions leading to the neutral signature of $\gamma[\boldsymbol{E}]$ is interesting in itself. Additionally, it can shed new light onto the theory of generally-relativistic spinors and the conformal $\mathrm{U}(2, 2)$-symmetry.

Keywords: field of complex linear frames on real differentiable manifold, Euler–Lagrange equations, semisimple Lie group, pseudo-Riemannian metric, neutral signature.

1. Introduction

In [14, 12] we presented a generally-covariant model of the field of complex linear frames $\boldsymbol{E}$ on a real manifold M of dimension n. The model is invariant under the natural action of the group $\mathrm{GL}(n, \mathbb{C})$. The joint assumption of the general covariance in the manifold M and the internal $\mathrm{GL}(n, \mathbb{C})$-invariance results in a very strong nonlinearity of the model. More precisely, Lagrangian density of $\boldsymbol{E}$ is given by

$$L = \sqrt{|\det[\gamma_{\mu\nu}]|}, \tag{1.1}$$

where $\gamma_{\mu\nu}$ are components of a 2-fold covariant complex tensor field on M, built algebraically of $\boldsymbol{E}$ and its first derivatives. The dependence on the derivatives is a low-order (second-order) polynomial. Such square root structure of (1.1) resembles the Born–Infeld electrodynamics [21]. Hence it may be called the modified Born–Infeld-type nonlinearity.

In a sense, our model [14, 12] is a complexification of tetrad approaches to gravitation proposed by Sławianowski [22–24] and developed by the author [8–11, 13]. Namely, in [22–24] Sławianowski suggested an alternative model of gravitation where geometrical (gravitational) degrees of freedom were described by components of the field of real linear frames $\boldsymbol{E}$ on a "space-time" manifold M of dimension n (the tetrad field if $n = 4$). The commonly applied theories of gravitation with the field of frames as a primary dynamical field (i.e. a field subject to the variational procedure), e.g. the Einstein theory formulated in the tetrad terms or the general metric-teleparallel theories [15, 18–20] are invariant under the natural action of pseudo-Euclidean subgroups $\mathrm{SO}(k, n-k)$ of $\mathrm{GL}(n, \mathbb{R})$ ($\mathrm{SO}(1,3)$ in the physical case where $n = 4$). Unlike this, the field of frames $\boldsymbol{E}$ in the approach presented by Sławianowski is ruled by $\mathrm{GL}(n, \mathbb{R})$ rather than by its subgroup $\mathrm{SO}(k, n-k)$. This extension of the internal symmetry group leads to a strong nonlinearity of the model. Namely, the Lagrangian density is of the Born–Infeld-type, i.e. of the form (1.1) with the real and symmetric $\gamma_{\mu\nu}$.

Incidentally, one deals with this kind of nonlinearity in the theory of t'Hooft-Polyakov monopole in p-brane models [3–5, 16, 17] and string models in the dynamics of multiplets of scalar fields [25] or the geometric theory of minimal surfaces [7].

We have generalized Sławianowski's ideas in [8–10]. Starting with Lagrangian of the form (1.1), we have constructed [8] a whole family of Lagrangians of $\boldsymbol{E}$ (see Eq. (3.19) in the present paper). Then we examined a tetrad approach to the gravitational field interacting with the bosonic matter [9, 10].

There is no canonical prescription for the Lagrangian density L_{total} of the field of real linear frames $\boldsymbol{E}$ (the gravitational field) interacting with a multiplet of complex scalar fields $\boldsymbol{\Phi}$ (the bosonic matter). Even the assumption that L_{total} is to be invariant under the natural action of $\mathrm{GL}(n, \mathbb{R})$ on the field of real frames $\boldsymbol{E}$ does not determine uniquely the Lagrangian density. However, two forms of L_{total} are privileged.

(a)

$$L_{\mathrm{total}} = f_{\mathrm{m}}\sqrt{|\det[\gamma_{\mu\nu}]|}, \tag{1.2}$$

where f_{m} denotes the Lagrangian function of the matter. Formula (1.2) and its natural generalization were proposed by Sławianowski [22, pp. 340–346 and 354]. Defining L_{total} by (1.2) means that one assumes the multiplicative decomposition of L_{total} into the "matter" part and the "gravitational" part, instead of the additive decomposition postulated in commonly applied theories of gravitation, e.g. in Einstein's general relativity,

$$(L_{\mathrm{total}})_{\mathrm{GR}} = (f_{\mathrm{m}} + R)\sqrt{|\det[g_{\mu\nu}]|},$$

where g is a pseudo-Riemannian metric on the space-time manifold and R denotes the curvature scalar for g. In a "matter-free" Universe, L_{total} (1.2) becomes simply (1.1) with $f_{\mathrm{m}} = 1$ (not $f_{\mathrm{m}} = 0$).

(b)

$$L_{\mathrm{total}} = \sqrt{|\det[G_{\mu\nu}]|}, \tag{1.3}$$

where $G_{\mu\nu}$ are components of a 2-fold covariant complex tensor field $G[\boldsymbol{E}, \boldsymbol{\Phi}]$ on M, built algebraically of $\boldsymbol{E}, \boldsymbol{\Phi}$ and their first derivatives. The natural choice of $G[\boldsymbol{E}, \boldsymbol{\Phi}]$ corresponds to Eqs. (3.20) and (3.21) in the present paper. (We have discussed it in [9].) Obviously, L_{total} (1.3) is of the Born–Infeld-type. From (3.21) it follows that, in the absence of matter ($\boldsymbol{\Phi} = 0$), L_{total} (1.3) becomes Lagrangian density (1.1), describing the "self-interacting" $\boldsymbol{E}$.

In [9, 10] we have chosen L_{total} (1.3) rather than (1.2). The reason? Even if the Born–Infeld-type nonlinearity of (1.3) is hardly treatable on the analytical level, it is relatively easy to find solutions to field equations derived from (1.3). So, in [9, 10] we present several families of solutions to these equations. On the other hand, the multiplicative decomposition of (1.2), breaking the Born–Infeld-type structure of L_{total}, results in field equations which are much more nonlinear, and so more hardly treatable on the level of analytical calculations than the equations derived from (1.3). Moreover, the field equations derived from (1.3) have a nice feature: they can be solved within the framework of the Lie groups theory. Strictly speaking, these equations correspond to the eigen problem for the Laplace–Beltrami operator on a semisimple Lie group. Consequently, for solutions to the field equations presented in [10], the components of $\boldsymbol{\Phi}$ are built of matrix elements of irreducible representations of a semisimple compact Lie group.

Let us go back to the case of a self-interacting field of real linear frames $\boldsymbol{E}$ (the absence of matter). In [11] we have analysed an example, interesting from the mathematical point of view, where the manifold M was the direct product of two semisimple Lie groups.

In [13] we investigated a case where

$$M = \mathbb{R} \times \mathrm{G}$$

and G was a semisimple Lie group. Solutions to the Euler–Lagrange equations for the field of frames $\boldsymbol{E}$, which we have found, are bases for the Lie algebra of left-invariant vector fields on $\mathbb{R} \times \mathrm{G}$ "deformed" by a $\mathrm{GL}(n, \mathbb{R})$-valued mapping of the exponential form.

In all cases concerning the field of real linear frames analysed by us, the Lagrangian density was of the Born–Infeld-type (Eq. (1.1) or its "small" modification) with the real and symmetric $\gamma_{\mu\nu}$ [8, 11, 13] or Hermitian $G_{\mu\nu}$ [9, 10] .

Although the field equations derived from Lagrangian of the form (1.1) are hardly treatable on the level of analytical calculations because of their very strong, essential, nonperturbative nonlinearity, this Lagrangian could stimulate interest per se. The main point is that this kind of the modified Born–Infeld-type nonlinearity is here a direct consequence of our symmetry demands. Namely, this structure is almost canonically implied by the joint assumption of the general covariance in space-time and the internal $\mathrm{GL}(n, \mathbb{R})$-invariance. Incidentally, one point must be stressed here. Obviously, Einstein's general relativity formulated in tetrad terms is internally invariant under the local ("x-dependent") action of $\mathrm{SO}(1, 3)$. The other tetrad models are globally invariant under this group. In the theory of generally-relativistic spinor fields, the matter Lagrangian is also invariant under local action of the structure group $SO(1,3)$. But it is less evident whether the purely gravitational part of Lagrangian (just the tetrad part) must be also local. There was even an idea of Borzeszkowski and Treder [1, 2] that some term globally, but not locally, invariant under $SO(1,3)$ might be perhaps responsible for dark matter. But if once we have admitted the global $SO(1,3)$-internal invariance of the gravitational Lagrangian, we must simply raise the question: "But why not the global $GL(4,\mathbb{R})$-internal invariance of the gravitational sector?" (The local one is evidently impossible if the tetrad alone is used as the gravitational potential.) In any case, the total group $GL(4,\mathbb{R})$ seems more natural. Consequently our idea of the $GL(n,\mathbb{R})$-internal invariance which taken jointly with the general covariance in space-time implies the modified Born–Infeld-type nonlinearity.

Let us go back to the model of the field of complex linear frames $\boldsymbol{E}$ with Lagrangian (1.1) [14, 12] quoted at the beginning of this section. As we mentioned above, the model is a complexification of tetrad approaches studied in the papers which we have just cited. This complexification means that we replace the real field of frames with the complex one. Consequently the internal symmetry group $GL(n,\mathbb{R})$ is replaced with $GL(n,\mathbb{C})$. Just as in the case of the real field of frames, the modified Born–Infeld-type nonlinearity for the field of complex linear frames (Eq. (1.1)) is the almost canonical feature, i.e. implied by the joint demand for the general covariance in the manifold M and the internal $GL(n,\mathbb{C})$-invariance.

Using the complex field of frames instead of the real one did not reduce computational difficulties resulting from nonlinearity of field equations. Moreover, it raised some additional questions. For example, we had to modify and adapt definitions of basic geometric objects built of the field of frames to the case where the field of frames is complex (Section 2 of [14]). Another problem concerned the tensorial (generally-covariant) form of field equations. Strictly speaking, it was related to the fact that, for the field of complex linear frames, the covariant differentiation corresponding to the teleparallelism connection was not uniquely defined. We analysed this problem in Section 3 of [14]. On the other hand, the complex field of frames also possesses some advantages. They were discussed in [14]. Here we quote one of them.

There is an essential difficulty in the standard approach to spinor structures. The point is that, unlike the tensor bundles describing bosonic fields, the $SL(2,\mathbb{C})$-ruled spinor bundles, that one uses in commonly accepted theories, are noncanonical, so to speak, "nonsoldered" to the space-time manifold. As it was shown in [26–29], substituting the gauge group $U(2,2)$ for $SL(2,\mathbb{C})$ may be a proper way to overcome problems which appear in standard $SL(2,\mathbb{C})$-based theories. Incidentally, the application of $U(2,2)$ to spinor structures is dicussed in details in [6, pp. 12–14 and 21–27].

Since the subgroup $SU(2,2)$ of $U(2,2)$ is the covering group of the Lorentz-conformal group and this last is not a subgroup of $GL(4,\mathbb{R})$, this substitution requires replacing the principal bundle of real linear frames with the structure group $GL(4,\mathbb{R})$ by another one. The natural choice is the bundle of complex linear frames with $GL(4,\mathbb{C})$ as the structure group. In other words, the complex tetrad field rather than the real one provides a proper framework underlying the U(2, 2)-based models of the spinor field interacting with the gravitational field. We shall go back to this question in Section 4 of the present paper.

2. Lagrangian of the field of complex linear frames. Solutions to Euler–Lagrange equations

In this section we shall review some concepts concerning the complex field of frames and the results which we have obtained in [14].

Let M be a real n-dimensional differentiable manifold of class C^∞. We denote by $\mathcal{F}(p, \mathbb{C})$ the set of all complex functions of class C^∞ defined in a neighbourhood of a point $p \in M$. Differentiability of $f \in \mathcal{F}(p, \mathbb{C})$ is understood as differentiability in the real domain. For $X, Y \in T_pM$, the complex tangent vector $X + iY$ at p is a $\mathbb{C}$-linear mapping $\mathcal{F}(p, \mathbb{C}) \to \mathbb{C}$ given by

$$(X + iY)f = Xf + iYf.$$

The complex tangent space $T_p(M, \mathbb{C})$ of M at p is the $\mathbb{C}$-linear space of all such vectors. The complex conjugate of $X + iY$ is

$$\overline{X + iY} = X - iY.$$

We denote by $T_p{}^*(M, \mathbb{C})$ the $\mathbb{C}-$dual space of $T_p(M, \mathbb{C})$. Each $\omega \in T_p{}^*(M, \mathbb{C})$ can be uniquely written as

$$\omega = \alpha + i\beta,$$

where $\alpha, \beta \in T_p{}^*M$. The complex conjugate of ω is

$$\overline{\omega} = \alpha - i\beta.$$

An ordered basis of $T_p(M, \mathbb{C})$ (resp. $T_p{}^*(M, \mathbb{C})$) is called the complex linear frame (resp. co-frame) at p. The principal bundle of complex linear frames (resp. co-frames) over M is denoted by $F(M, \mathbb{C})$ (resp. $F^*(M, \mathbb{C})$). The structure group of $F(M, \mathbb{C})$ and $F^*(M, \mathbb{C})$ is $\mathrm{GL}(n, \mathbb{C})$.

Let

$$\boldsymbol{E} = (E_A) : M \to F(M, \mathbb{C}), \qquad A = 0, 1, \ldots, n-1,$$

be a cross section of class C^∞. In other words, $\boldsymbol{E}$ is a field of complex linear frames on M. Let

$$\widetilde{\boldsymbol{E}} = (E^A) : M \to F^*(M, \mathbb{C})$$

be a field of complex linear co-frames $\mathbb{C}$-dual to $\boldsymbol{E}$, i.e.

$$\langle E^A, E_B \rangle = \delta^A{}_B.$$

The structure group $\mathrm{GL}(n, \mathbb{C})$ of the bundle $F(M, \mathbb{C})$ can act on $\boldsymbol{E}$ as follows,

$$\boldsymbol{E} = (E_A) \longmapsto \boldsymbol{E}L = ((\boldsymbol{E}L)_A) = (L^B{}_A E_B), \tag{2.1}$$

where $L = [L^A{}_B] \in \mathrm{GL}(n, \mathbb{C})$.

We define a system of complex functions $\gamma^{A}{}_{B\overline{C}}[\boldsymbol{E}]$, $A, B, \overline{C} = 0, 1, \ldots, n-1$, of class C^{∞} on M by

$$[E_B, \overline{E_C}] = \gamma^{A}{}_{B\overline{C}}[\boldsymbol{E}]\ E_A. \tag{2.2}$$

Using (2.1) gives

$$\gamma^{A}{}_{B\overline{C}}[\boldsymbol{E}L] = (L^{-1})^{A}{}_{D} L^{F}{}_{B} \overline{L^{G}{}_{C}}\ \gamma^{D}{}_{F\overline{G}}[\boldsymbol{E}], \tag{2.3}$$

where $L \in \mathrm{GL}(n, \mathbb{C})$ and L^{-1} is the inverse matrix of L.

REMARK 2.1. Numerically, the values of the indices C in E_C and $\overline{C}$ in $\gamma^{A}{}_{B\overline{C}}$ are the same. The bar over the index in $\gamma^{A}{}_{B\overline{C}}$ simply indicates that, under the action of $\mathrm{GL}(n, \mathbb{C})$, the two indices, i.e. C and $\overline{C}$, are subject to two different transformation rules. For C in E_C, it is the rule (2.1), whereas the index $\overline{C}$ (in $\gamma^{A}{}_{B\overline{C}}$) is related to the complex conjugate $\overline{L^{G}{}_{C}}$ of $L^{G}{}_{C}$ in Eq. (2.3).

From the quantities $\gamma^{A}{}_{B\overline{C}}[\boldsymbol{E}]$ we build a complex tensor field $S[\boldsymbol{E}]$ of type (1, 2) and a complex covariant tensor field $\gamma[\boldsymbol{E}]$ of degree 2, namely

$$S[\boldsymbol{E}] = \frac{1}{2}\gamma^{A}{}_{B\overline{C}}[\boldsymbol{E}]\ E_A \otimes E^B \otimes \overline{E^C} \tag{2.4}$$

and

$$\gamma[\boldsymbol{E}] = \gamma_{\ \overline{A}B}[\boldsymbol{E}]\ \overline{E^A} \otimes E^B, \tag{2.5}$$

$$\gamma_{\ \overline{A}B}[\boldsymbol{E}] = \overline{\gamma^{C}{}_{A\overline{D}}[\boldsymbol{E}]}\gamma^{D}{}_{B\overline{C}}[\boldsymbol{E}]. \tag{2.6}$$

For the pairs of indices: $(A, \overline{A})$, $(C, \overline{C})$ and $(D, \overline{D})$, see Remark 2.1.

Since $\boldsymbol{E} = (E_A)$ is a cross section of $F(M, \mathbb{C})$, so is its complex conjugate $\overline{\boldsymbol{E}} = (\overline{E_A})$. Hence $\boldsymbol{E}$ induces two flat linear connections in $F(M, \mathbb{C})$, denoted by $\Gamma_{\mathrm{tel}}[\boldsymbol{E}]$ and $\Gamma_{\mathrm{tel}}[\overline{\boldsymbol{E}}]$ — called the teleparallelism connections. They are uniquely determined by the conditions

$${}^{E}\nabla E_A = 0 \qquad \text{and} \qquad {}^{\overline{E}}\nabla(\overline{E_A}) = 0, \tag{2.7}$$

$A = 0, 1, \ldots, n-1$, where ${}^{E}\nabla$ (resp. ${}^{\overline{E}}\nabla$) denotes the covariant differentiation corresponding to $\Gamma_{\mathrm{tel}}[\boldsymbol{E}]$ (resp. $\Gamma_{\mathrm{tel}}[\overline{\boldsymbol{E}}]$).

In terms of a local coordinate system (x^{μ}), $\mu = 0, 1, \ldots, n-1$, in a coordinate neighbourhood $U \subset M$,

$$\begin{aligned} S[\boldsymbol{E}]|U &= S^{\lambda}{}_{\mu\nu}[\boldsymbol{E}]\frac{\partial}{\partial x^{\lambda}} \otimes dx^{\mu} \otimes dx^{\nu}, \\ S^{\lambda}{}_{\mu\nu}[\boldsymbol{E}] &= \frac{1}{2}(\Gamma^{\lambda}{}_{\mu\nu}[\boldsymbol{E}] - \overline{\Gamma^{\lambda}{}_{\nu\mu}[\boldsymbol{E}]}) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \gamma[\boldsymbol{E}]|U &= \gamma_{\mu\nu}[\boldsymbol{E}]\ dx^{\mu} \otimes dx^{\nu}, \\ \gamma_{\mu\nu}[\boldsymbol{E}] &= 4\overline{S^{\lambda}{}_{\mu\rho}[\boldsymbol{E}]}S^{\rho}{}_{\nu\lambda}[\boldsymbol{E}], \end{aligned} \tag{2.9}$$

where the components of $\Gamma_{\mathrm{tel}}[\boldsymbol{E}]$ and $\Gamma_{\mathrm{tel}}[\overline{\boldsymbol{E}}]$ with respect to (x^{μ}) have the form

$$\Gamma^{\lambda}{}_{\mu\nu}[\boldsymbol{E}] = E_A{}^{\lambda} \frac{\partial}{\partial x^{\nu}} E^A{}_{\mu}, \qquad \Gamma^{\lambda}{}_{\mu\nu}[\overline{\boldsymbol{E}}] = \overline{\Gamma^{\lambda}{}_{\mu\nu}[\boldsymbol{E}]} \tag{2.10}$$

and $E_A{}^{\lambda}$ (resp. $E^A{}_{\mu}$) are the components of a vector field E_A (resp. a 1-form E^A) with respect to (x^{μ}).

Eqs. (2.6) and (2.9) imply that the matrices $[\gamma_{\ \overline{A}B}[\boldsymbol{E}]]$ and $[\gamma_{\mu\nu}[\boldsymbol{E}]]$ are Hermitian:

$$\overline{\gamma_{\ \overline{A}B}[\boldsymbol{E}]} = \gamma_{\ \overline{B}A}[\boldsymbol{E}], \qquad \overline{\gamma_{\mu\nu}[\boldsymbol{E}]} = \gamma_{\nu\mu}[\boldsymbol{E}]. \tag{2.11}$$

By virtue of (2.8), $S^{\lambda}{}_{\mu\nu}[\boldsymbol{E}]$ are skew-Hermitian with respect to the lower indices,

$$\overline{S^{\lambda}{}_{\mu\nu}[\boldsymbol{E}]} = -S^{\lambda}{}_{\nu\mu}[\boldsymbol{E}]. \tag{2.12}$$

The covariant complex tensor field $\gamma[\boldsymbol{E}]$ of degree 2 induces a scalar density of weight 1 on M, denoted by $\sqrt{|\det\gamma[\boldsymbol{E}]|}$. In terms of a local coordinate system (x^{μ}), $\sqrt{|\det\gamma[\boldsymbol{E}]|}$ is represented by a function

$$\sqrt{|\det[\gamma_{\mu\nu}[\boldsymbol{E}]]|}, \tag{2.13}$$

where $\gamma_{\mu\nu}[\boldsymbol{E}]$ are given by (2.9).

A Lagrangian $\mathcal{L}[\boldsymbol{E}]$ of the field of complex linear frames $\boldsymbol{E}$ is a real differential n-form on M corresponding to $\sqrt{|\det\gamma[\boldsymbol{E}]|}$. More precisely, in terms of a local coordinate system (x^{μ}) in a coordinate neighbourhood $U \subset M$,

$$\mathcal{L}[\boldsymbol{E}]|U = \sqrt{|\det[\gamma_{\mu\nu}[\boldsymbol{E}]]|}\, dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{n-1}. \tag{2.14}$$

$\mathcal{L}[\boldsymbol{E}]$ is built algebraically of $\boldsymbol{E}, \overline{\boldsymbol{E}}$ and their first derivatives (see (2.9) and (2.10)). Consequently, field equations derived from $\mathcal{L}[\boldsymbol{E}]$ are differential equations of the second order.

Combining (2.1) and (2.3) gives

$$\mathcal{L}[\boldsymbol{E}L] = \mathcal{L}[\boldsymbol{E}] \tag{2.15}$$

for $L \in \mathrm{GL}(n,\mathbb{C})$. Thus the Lagrangian $\mathcal{L}[\boldsymbol{E}]$ is invariant under the global action of $\mathrm{GL}(n,\mathbb{C})$.

Let Diff$M$ be the group of all diffeomorphisms of M onto itself of class C^{∞}. For $\varphi \in \mathrm{Diff}M$, we denote by $\widetilde{\varphi}_*$ an automorphism of the tensor algebra of all complex tensor fields on M induced by φ_*. We set

$$\varphi_*\boldsymbol{E} = (\varphi_* E_A), \qquad A = 0, 1, \ldots, n-1.$$

From the definition of $\gamma^A{}_{B\overline{C}}[\boldsymbol{E}]$ (Eq. (2.2)), it follows that

$$\mathcal{L}[\varphi_*\boldsymbol{E}] = \tilde{\varphi}_*(\mathcal{L}[\boldsymbol{E}]) \tag{2.16}$$

for $\varphi \in \mathrm{Diff}M$. Eq. (2.16) means that the Lagrangian $\mathcal{L}[\boldsymbol{E}]$ is Diff$M$-covariant (generally-covariant).

To avoid the multitude of symbols, we shall denote $\gamma^A{}_{B\overline{C}}[\boldsymbol{E}]$, $\gamma_{\mu\nu}[\boldsymbol{E}]$, $\Gamma^{\lambda}{}_{\mu\nu}[\boldsymbol{E}]$, … simply by $\gamma^A{}_{B\overline{C}}, \gamma_{\mu\nu}$, $\Gamma^{\lambda}{}_{\mu\nu}, \ldots$, omitting the label $\boldsymbol{E}$, unless otherwise stated.

Let us fix a coordinate neighbourhood $U \subset M$ with a local coordinate system (x^μ) and denote

$$\det\gamma = \det[\gamma_{\mu\nu}]. \tag{2.17}$$

The Euler–Lagrange equations derived from the Lagrangian $\mathcal{L}[\boldsymbol{E}]$ may be concisely written as follows:

$$\frac{\partial}{\partial x^\beta}\mathcal{H}_A{}^{\alpha\beta} + \mathcal{J}_A{}^\alpha = 0, \qquad A, \alpha = 0, 1, \ldots, n-1, \tag{2.18}$$

where

$$\begin{aligned}\mathcal{J}_A{}^\alpha &= \sqrt{|\det\gamma|}E_A{}^\lambda\gamma^{\mu\nu}(\Gamma^\alpha{}_{\mu\rho}\overline{S^\rho{}_{\nu\lambda}} - \Gamma^\alpha{}_{\rho\nu}S^\rho{}_{\mu\lambda}),\\ \mathcal{H}_A{}^{\alpha\beta} &= E_A{}^\lambda H_\lambda{}^{\alpha\beta},\\ H_\lambda{}^{\alpha\beta} &= \sqrt{|\det\gamma|}(\gamma^{\alpha\mu}\overline{S^\beta{}_{\mu\lambda}} - \gamma^{\mu\beta}S^\alpha{}_{\mu\lambda})\end{aligned} \tag{2.19}$$

and $[\gamma^{\mu\nu}]$ denotes the inverse matrix of $[\gamma_{\mu\nu}]$.

Since $\sqrt{|\det\gamma|}$ is a scalar density of weight 1 on M, the formula (2.19) implies that $H_\lambda{}^{\alpha\beta}$ are components (with respect to a local coordinate system (x^μ)) of a complex tensor density of type (2, 1) and weight 1 on M. We denote it by $H[\boldsymbol{E}]$.

To rewrite the primary Euler–Lagrange equations (2.18) in explicitly tensorial (generally-covariant) form, we need a covariant differential $\nabla H[\boldsymbol{E}]$ of $H[\boldsymbol{E}]$. Since $\boldsymbol{E}$ induces the two teleparallelism connections, $\Gamma_{\text{tel}}[\boldsymbol{E}]$ and $\Gamma_{\text{tel}}[\overline{\boldsymbol{E}}]$ (see (2.7) and (2.10)), one can define $\nabla H[\boldsymbol{E}]$ in various ways. In [14] we imposed additional conditions which determine uniquely $\nabla H[\boldsymbol{E}]$ (see Section 3 of [14]). The covariant differential $\nabla H[\boldsymbol{E}]$ that we obtained in this manner is given by

$$\nabla_\rho H_\lambda{}^{\alpha\beta} = \frac{\partial}{\partial x^\rho}H_\lambda{}^{\alpha\beta} - \frac{1}{2}(\Gamma^\omega{}_{\omega\rho} + \overline{\Gamma^\omega{}_{\omega\rho}})H_\lambda{}^{\alpha\beta} - \Gamma^\omega{}_{\lambda\rho}H_\omega{}^{\alpha\beta} + \Gamma^\alpha{}_{\omega\rho}H_\lambda{}^{\omega\beta} + \overline{\Gamma^\beta{}_{\omega\rho}}H_\lambda{}^{\alpha\omega}. \tag{2.20}$$

Using (2.20), one can rewrite the Euler–Lagrange equations (2.18) in the explicitly tensorial form:

$$\nabla_\beta H_\lambda{}^{\alpha\beta} + S^\omega{}_{\omega\beta}H_\lambda{}^{\alpha\beta} + \overline{\Xi^\omega{}_{\omega\beta}}H_\lambda{}^{\alpha\beta} = 0, \qquad \alpha, \lambda = 0, 1, \ldots, n-1, \tag{2.21}$$

where

$$\Xi^\lambda{}_{\mu\nu}[\boldsymbol{E}] = \frac{1}{2}(\Gamma^\lambda{}_{\mu\nu}[\boldsymbol{E}] - \Gamma^\lambda{}_{\nu\mu}[\boldsymbol{E}]).$$

Obviously, Eqs. (2.21) are local, i.e. they are valid in a coordinate neighbourhood $U \subset M$ only.

We define a system of complex functions $H_A{}^{B\overline{C}}$, $A, B, \overline{C} = 0, 1, \ldots, n-1$, by

$$H_A{}^{B\overline{C}} = \frac{1}{2}\sqrt{|\operatorname{Det}\gamma|}(\gamma^{B\overline{F}}\ \ \overline{\gamma^C{}_{F\overline{A}}}\ \ - \langle\overline{E^G}, E_A\rangle\gamma^{F\overline{C}}\gamma^B{}_{F\overline{G}}), \tag{2.22}$$

where

$$\operatorname{Det}\gamma = \det[\gamma_{\overline{A}B}], \tag{2.23}$$

$[\gamma^{B\overline{F}}]$ is the inverse matrix of $[\gamma\ _{\overline{B}F}]$ and $\langle\overline{E^G}, E_A\rangle$ denotes the value of a 1-form $\overline{E^G}$ on a vector field E_A.

REMARK 2.2. The capital D in $\mathrm{Det}\,\gamma$ is used to distinguish between the determinants of matrices $[\gamma\ _{\overline{A}B}]$ with capital indices and $[\gamma_{\mu\nu}]$ with the small ones (cf. (2.17)).

A straightforward calculation turns the local field equations (2.21) into a system of global equations, without explicit use of coordinates, which is valid all over M:

$$\overline{E_C} H_A{}^{B\overline{C}} + \gamma^D{}_{D\overline{C}} H_A{}^{B\overline{C}} - \frac{1}{2}\langle\overline{E^F}, E_D\rangle H_A{}^{B\overline{C}}\ \overline{E_C}(\langle E^D, \overline{E_F}\rangle) = 0, \qquad A, B = 0, 1, \ldots, n-1. \tag{2.24}$$

We assume that

$$M = \mathbb{R} \times \mathrm{G}, \tag{2.25}$$

where G is a real $(n-1)$-dimensional semisimple Lie group and $\mathbb{R}$ is regarded as a 1-dimensional Lie group of translations.

Let $(\mathcal{E}_\Lambda)$, $\Lambda = 1, \ldots, n-1$, be a basis for the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G.

REMARK 2.3. From now on the Greek capital indices $(\Delta, \overline{\Lambda}, \Sigma, \ldots)$ will range from 1 to $n-1$ $(n = \dim M)$. Obviously, the Latin capital indices $(A, \overline{B}, C, \ldots)$ will still range from 0 to $n-1$.

The components of the Killing–Cartan form K of $\mathfrak{g}$ with respect to $(\mathcal{E}_\Lambda)$ are given by

$$K_{\Lambda\Sigma} = C^\Delta{}_{\Lambda\Pi}\ C^\Pi{}_{\Sigma\Delta}, \tag{2.26}$$

where $C^\Delta{}_{\Lambda\Pi}$ are the structure constants of $\mathfrak{g}$,

$$[\mathcal{E}_\Lambda, \mathcal{E}_\Sigma] = C^\Delta{}_{\Lambda\Sigma}\ \mathcal{E}_\Delta. \tag{2.27}$$

We define a vector field $\mathcal{E}_0$ along $\mathbb{R}$ by

$$\mathcal{E}_0 f = \dot{f} \tag{2.28}$$

for any function $f : \mathbb{R} \to \mathbb{C}$ of class C^∞, where $\dot{f}$ denotes the "ordinary" derivative of f.

Now, we shall deform the vector fields $\mathcal{E}_A$ by a matrix-valued mapping of the exponential form. Namely, we define the mappings

$$\lambda : \mathbb{R} \longrightarrow \mathbb{C} - \{0\} \quad \text{and} \quad A = [A^\Lambda{}_\Sigma] : \mathbb{R} \longrightarrow \mathrm{GL}(n-1, \mathbb{C})$$

of class C^∞ by

$$\lambda(t) = Ce^{(\alpha+i\beta)t}, \tag{2.29}$$

$$A(t) = \exp(t(\kappa + i\omega)X) \quad \text{for } t \in \mathbb{R}, \tag{2.30}$$

where

$$X \in M_{(n-1)\times(n-1)}(\mathbb{R})$$

and $\alpha, \beta, \kappa, \omega, C$ are real constants such that

$$C > 0 \quad \text{and} \quad \kappa^2 + \omega^2 > 0. \tag{2.31}$$

Using the vector fields $\mathcal{E}_A$, the function λ and the components $A^\Lambda{}_\Sigma$ of the mapping A, we construct a system of complex vector fields E_A, $A = 0, 1, \ldots, n-1$, on $M = \mathbb{R}\times\mathrm{G}$ according to the prescription

$$E_0 = \lambda \ \mathcal{E}_0, \quad E_\Sigma = A^\Lambda{}_\Sigma \ \mathcal{E}_\Lambda, \tag{2.32}$$

i.e.

$$(E_\Sigma)_{(t,g)} = A^\Lambda{}_\Sigma(t)(\mathcal{E}_\Lambda)_g \quad \text{and} \quad (E_0)_{(t,g)} = \lambda(t)(\mathcal{E}_0)_t \text{ for } (t,g) \in M = \mathbb{R} \times \mathrm{G}.$$

Since λ does not vanish anywhere and $A = [A^\Lambda{}_\Sigma]$ is $\mathrm{GL}(n-1, \mathbb{C})$-valued, the vector fields E_A form a field of complex linear frames on $M = \mathbb{R} \times \mathrm{G}$. We shall denote it by $\boldsymbol{E}$.

THEOREM 2.1. *Suppose that the matrix X in Eq.* (2.30) *is K-symmetric, i.e.*

$$K_{\Lambda\Pi} X^\Pi{}_\Sigma = K_{\Sigma\Pi} X^\Pi{}_\Lambda, \tag{2.33}$$

where

$$[X^\Lambda{}_\Sigma] = X.$$

In addition, let us assume that

$$\mathrm{tr}(X^2) \neq -\frac{4\beta^2}{\kappa^2 + \omega^2}. \tag{2.34}$$

Let the complex vector fields E_A be defined by (2.32), (2.30) *and* (2.29). *Then* $\boldsymbol{E} = (E_A)$ *is a solution to field equations* (2.24).

THEOREM 2.2. *Let $\boldsymbol{E} = (E_A)$ be a solution to the field equations given by Theorem* 2.1. *Then the twice covariant tensor field $\gamma[\boldsymbol{E}]$, defined by* (2.5) *and* (2.6), *is real and is a pseudo-Riemannian metric on $M = \mathbb{R} \times \mathrm{G}$.*

For the proofs (of Theorems 2.1 and 2.2), see [14, pp.130–133].

3. Solutions to field equations with the neutral signature of $\gamma[\boldsymbol{E}]$

In this section we shall analyse a special case of that given by (2.25), where

$$\dim M = 4$$

and

$$M = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R}), \tag{3.1}$$

i.e.

$$\mathrm{G} = \mathrm{SL}(2, \mathbb{R}).$$

The Lie algebra $\mathfrak{g}$ of left-invariant vector fields on $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of 2×2 real traceless matrices. Thus the Killing-Cartan form K of $\mathfrak{g}$ possesses the signature

$$(+ + -).$$

We choose a basis $(\mathcal{E}_\Lambda)$, $\Lambda = 1, 2, 3$ $(n-1=3)$, for the Lie algebra $\mathfrak{g}$ such that the components of K are given by

$$[K_{\Lambda\Sigma}] = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix}. \tag{3.2}$$

In [14] we analysed another case corresponding to $\dim M = 4$,

$$M = \mathbb{R} \times \mathrm{SU}(2).$$

Then condition (2.33) in Theorem 2.1 means that matrix X is simply symmetric. Consequently,

$$\mathrm{tr}(X^2) > 0 \tag{3.3}$$

if $X \neq 0$. By virtue of (3.3), for any solution $\boldsymbol{E}$ to the field equations given by Theorem 2.1, the pseudo-Riemannian metric $\gamma[\boldsymbol{E}]$ possesses the normal-hyperbolic signature $(+---)$. (For details, see [14, pp. 133-134].)

Let us go back to the case (3.1), i.e.

$$M = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R}).$$

Then condition (2.33) (in Theorem 2.1) admits the matrices

$$X \in M_{3\times 3}(\mathbb{R})$$

such that

$$\mathrm{tr}(X^2) < 0. \tag{3.4}$$

Indeed, let

$$X = [X^\Lambda{}_\Sigma] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -k \\ 0 & k & 0 \end{bmatrix}, \tag{3.5}$$

where $k \in \mathbb{R}$ and $k \neq 0$. Denoting

$$X_{\Lambda\Sigma} = K_{\Lambda\Pi} X^\Pi{}_\Sigma \tag{3.6}$$

and using (3.2), we obtain

$$[X_{\Lambda\Sigma}] = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 0 & k & 0 \end{bmatrix}.$$

Thus

$$X_{\Lambda\Sigma} = X_{\Sigma\Lambda}, \tag{3.7}$$

where by virtue of (3.6), Eq. (3.7) coincides with (2.33).

Since the matrix $X = [X^\Lambda{}_\Sigma]$ is real,

$$\mathrm{tr}(X^T X) > 0. \tag{3.8}$$

By (3.5),

$$X^T = -X.$$

Hence, condition (3.8) becomes

$$\operatorname{tr}(X^2) < 0.$$

We have

$$\operatorname{tr}(X^2) = -k^2. \tag{3.9}$$

THEOREM 3.1. *Assume that condition* (3.1) *holds true. Then there exist solutions to field equations* (2.24) *with the pseudo-Riemannian metric* $\gamma[\boldsymbol{E}]$ *endowed with the neutral signature*

$$(++--).$$

Proof: Let $\boldsymbol{E} = (E_A)$ be a solution to field equations (2.24) given by Theorem 2.1. Let k be a real constant such that

$$k^2 > \frac{4\beta^2}{\kappa^2 + \omega^2} \tag{3.10}$$

(see (2.31)). In addition, we assume that

$$\operatorname{tr}(X^2) = -k^2, \tag{3.11}$$

where X is the matrix in Eq. (2.30). For example, X given by (3.5) satisfies simultaneously condition (3.11) and assumption (2.33) of Theorem 2.1. By (3.10), assumption (2.34) is also satisfied.

We have (cf. [14, p.133])

$$\gamma[\boldsymbol{E}] = \mathcal{K}_{AB}\ \mathcal{E}^A \otimes \mathcal{E}^B,$$
$$[\mathcal{K}_{AB}]_{4\times 4} = \left[\begin{array}{c|c} 4\beta^2 + (\kappa^2+\omega^2)\operatorname{tr}(X^2) & 0 \\ \hline 0 & \widehat{K} \end{array}\right], \tag{3.12}$$

where

$$\widehat{K} = [K_{\Lambda\Sigma}]_{3\times 3}$$

and $(\mathcal{E}^A)$ denotes a field of linear co-frames dual to $\boldsymbol{\mathcal{E}} = (\mathcal{E}_A)$ (for $\mathcal{E}_0$ and $\mathcal{E}_\Lambda$ see (2.28) and (3.2), respectively). Since the vector fields $\mathcal{E}_A$ are real, the 1-forms $\mathcal{E}^A$ are also real.

Substituting (3.11) and (3.2) into (3.12) yields

$$[\mathcal{K}_{AB}] = \begin{bmatrix} 4\beta^2 - (\kappa^2+\omega^2)k^2 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}. \tag{3.13}$$

By (3.10),

$$4\beta^2 - (\kappa^2+\omega^2)k^2 < 0. \tag{3.14}$$

Finally, combining (3.13), (3.14) and the fact that the 1-forms $\mathcal{E}^A$ are real, we find that the signature of $\gamma[\boldsymbol{E}]$ is

$$(-++-), \quad \text{i.e.} \quad (++--). \qquad \square$$

It is easy to verify that in the case

$$M = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R}),$$

considered in Theorem 3.1, not only

$$\mathrm{tr}(X^2) < 0$$

but also

$$\mathrm{tr}(X^2) > 0$$

are admitted by assumption (2.33). Consequently, apart from solutions to the field equations leading to the neutral signature $(++--)$ of $\gamma[\boldsymbol{E}]$, there exist solutions for which the signature of $\gamma[\boldsymbol{E}]$ is normal-hyperbolic

$$(+++-).$$

As we mentioned above, in the case

$$M = \mathbb{R} \times \mathrm{SU}(2), \tag{3.15}$$

$\gamma[\boldsymbol{E}]$ possesses the normal-hyperbolic signature

$$(+---)$$

for all solutions to the field equations given by Theorem 2.1.

Finally, generalizing (3.1) and (3.15), we go back to the case (2.25), i.e.

$$\dim M = 4 \quad \text{and} \quad M = \mathbb{R} \times \mathrm{G}, \tag{3.16}$$

where G is a real 3-dimensional semisimple Lie group. Then the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G is isomorphic to $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$. Consequently, we obtain the following result.

COROLLARY 3.1. *Let condition* (3.16) *be satisfied. Taking all solutions to the field equations given by Theorem* 2.1, *we obtain the following signatures of* $\gamma[\boldsymbol{E}]$*:*

(a) *normal-hyperbolic*

$$(+---) \qquad \text{or} \qquad (+++-),$$

(b) *neutral*

$$(++--).$$

The result of Corollary 3.1 is interesting from the mathematical point of view. Namely, existence of solutions (to the field equations) leading to the neutral signature $(++--)$ of $\gamma[\boldsymbol{E}]$ differentiates the case where the tetrad field is complex from that concerning the real tetrad field.

As we mentioned in Section 1, the models of the field of real linear frames $\boldsymbol{E}$ with Lagrangian of the Born–Infeld-type are investigated in [22–24] and [8, 11, 13]. For the real field of frames, Eqs. (2.2), (2.4), (2.5) and (2.6) become

$$[E_B, E_C] = \gamma^A{}_{BC} E_A,$$

$$S[\boldsymbol{E}] = \frac{1}{2}\gamma^A{}_{BC} \;\; E_A \otimes E^B \otimes E^C$$

and

$$\gamma[\boldsymbol{E}] = \gamma_{AB} E^A \otimes E^B, \qquad \gamma_{AB} = \gamma^C{}_{AD}\gamma^D{}_{BC}. \tag{3.17}$$

The Lagrangian density is given by

$$L = \sqrt{|\det[\gamma_{\mu\nu}]|} \tag{3.18}$$

or

$$L = f[S]\sqrt{|\det[\gamma_{\mu\nu}]|}, \tag{3.19}$$

where $\gamma_{\mu\nu}$ are the components of $\gamma[\boldsymbol{E}]$ with respect to a local coordinate system and

$$f[S] : M \longrightarrow \mathbb{R}$$

is a function built algebraically of the components of $S[\boldsymbol{E}]$. In the "physical case" corresponding to (3.16), for all solutions to field equations which Sławianowski and the author found, the signature of $\gamma[\boldsymbol{E}]$ is normal-hyperbolic $(+---)$ or $(+++-)$. Solutions with the neutral signature have not been found yet.

In [9, 10] we proposed a model of the field of real frames $\boldsymbol{E}$ interacting with a multiplet of complex scalar fields $\boldsymbol{\Phi} = (\Phi^a)$, i.e.

$$\boldsymbol{\Phi} = (\Phi^a) : M \longrightarrow \mathbb{C}^k, \qquad a = 1, 2, \ldots, k.$$

In this approach the real $\gamma[\boldsymbol{E}]$ (Eqs. (3.17)) is replaced with a complex tensor field $G[\boldsymbol{E}, \boldsymbol{\Phi}]$ of type (0,2) defined as follows:

$$G[\boldsymbol{E}, \boldsymbol{\Phi}] = G_{AB} E^A \otimes E^B,$$
$$G_{AB} = (1 - c_1 \eta_{\overline{a}b} \overline{\Phi^a} \Phi^b)\gamma_{AB} + c_2 \eta_{\overline{a}b} \overline{(E_A \Phi^a)} E_B \Phi^b, \tag{3.20}$$

where $c_1 > 0$ and $c_2 > 0$ are constants, γ_{AB} are given by (3.17),

$$\eta_{\overline{a}b} = \eta(i_a, i_b),$$

η is a nondegenerate Hermitian form on $\mathbb{C}^k$ and (i_a), $a = 1, 2, \ldots, k$, is the standard basis for $\mathbb{C}^k$. In terms of a local coordinate system (x^μ) in a coordinate neighbourhood $U \subset M$ we have

$$G[\boldsymbol{E}, \boldsymbol{\Phi}]|U = G_{\mu\nu} dx^\mu \otimes dx^\nu,$$
$$G_{\mu\nu} = (1 - c_1 \eta_{\overline{a}b} \overline{\Phi^a} \Phi^b)\gamma_{\mu\nu} + c_2 \eta_{\overline{a}b} \overline{\left(\frac{\partial}{\partial x^\mu} \Phi^a\right)} \frac{\partial}{\partial x^\nu} \Phi^b, \tag{3.21}$$

where $\gamma_{\mu\nu}$ are the components of $\gamma[\boldsymbol{E}]$ with respect to (x^μ).

The Lagrangian density of the system $(\boldsymbol{E}, \boldsymbol{\Phi})$ is given by

$$L_{\text{total}} = \sqrt{|\det[G_{\mu\nu}]|}. \tag{3.22}$$

Obviously, it is of the Born–Infeld-type.

In the "physical case" (3.16), for all solutions to the Euler–Lagrange equations derived from (3.22) which we have found, $G[\boldsymbol{E}, \boldsymbol{\Phi}]$ becomes a real pseudo-Riemannian metric on $M = \mathbb{R} \times \mathrm{G}$. Moreover, just as for the model of self-interacting field of real frames quoted above ([22–24] and [8, 11, 13]), the signature of $G[\boldsymbol{E}, \boldsymbol{\Phi}]$ is normal-hyperbolic $(+---)$ or $(+++-)$.

Summarizing, the existence of solutions with the neutral signature of $\gamma[\boldsymbol{E}]$ is an intrinsic feature of the model of the field of complex linear frames investigated in our present paper which differentiates this model from those with the field of real frames.

These solutions with the neutral signature are also interesting from the physical point of view. Namely, they can shed new light onto the theory of generally-relativistic spinors and the conformal U(2, 2)-symmetry. We shall discuss it in the next section.

4. U(2, 2)-symmetry. The status of space-time metric

In the U(2, 2)-ruled model of the bispinor field interacting with the gravitational field, proposed by Sławianowski [26–29] and mentioned in Section 1 of the present paper, the space $\mathbb{C}^4$ of values of the bispinor field is endowed with a Hermitian form Δ of the neutral signature $(++--)$. The pseudo-unitary group $\mathrm{U}(4, \Delta)$ consists of all $\mathbb{C}$-linear transformations of $\mathbb{C}^4$ preserving Δ. Obviously, it is isomorphic to U(2, 2). The particular choice of Δ is physically not essential. It is only the signature of Δ that matters. Let us only recall two commonly used forms of Δ, corresponding to the standard $\mathrm{SL}(2, \mathbb{C})$-based spinor structures (Dirac and Weyl–Van der Waerden representations).

(a)

$$[\Delta_{\overline{A}B}] = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}, \tag{4.1}$$

where $[\Delta_{\overline{A}B}]$ is the matrix of Δ with respect to the standard basis (k_A), $A = 1, \ldots, 4$, for $\mathbb{C}^4$,

$$\Delta_{\overline{A}B} = \Delta(k_A, k_B).$$

Then

$$\mathrm{U}(4, \Delta) = \mathrm{U}(2, 2).$$

(b) Identifying $\mathbb{C}^4$ with the direct sum $\mathbb{C}^2 \times \overline{\mathbb{C}^2}^*$, where $\overline{\mathbb{C}^2}^*$ denotes the space of antilinear functions on $\mathbb{C}^2$, one can define Δ in the "canonical" manner,

$$\Delta((u_1, v_1), (u_2, v_2)) = \overline{v_1(u_2)} + v_2(u_1) \tag{4.2}$$

for $(u_1, v_1), (u_2, v_2) \in \mathbb{C}^4 = \mathbb{C}^2 \times \overline{\mathbb{C}^2}^*$, i.e. $u_1, u_2 \in \mathbb{C}^2$ and $v_1, v_2 \in \overline{\mathbb{C}^2}^*$. Then

$$[\Delta_{\overline{A}B}] = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \tag{4.3}$$

where I_2 denotes the 2×2 identity matrix. Obviously, the standard basis (k_1, k_2, k_3, k_4) for $\mathbb{C}^4$ is identified with $(i_1, i_2, j^{*1}, j^{*2})$, where (i_1, i_2) denotes the standard basis for $\mathbb{C}^2$ and (j^{*1}, j^{*2}) is the basis for $\overline{\mathbb{C}^2}^*$, antidual to (i_1, i_2).

In Sławianowski's model [26–29] the geometrodynamical sector is described by the following independent dynamical variables:

(a) A pseudo-Riemannian metric g on a 4-dimensional manifold M;

(b) A U(4, Δ)-ruled connection form A, i.e. a one-form A on M with values in the Lie algebra $\mathfrak{u}(4, \Delta)$ of U(4, Δ),

$$A: \quad M \ni p \longrightarrow A_p \in L(T_pM, \mathfrak{u}(4, \Delta)),$$

which is subject to the following transformation rule

$$A \mapsto A' = (L^{-1})AL + (L^{-1})dL \tag{4.4}$$

for any mapping

$$L : M \longrightarrow \mathrm{U}(4, \Delta) \tag{4.5}$$

of class C^∞, where $(L^{-1})(p)$ is the inverse matrix of $L(p)$ $(p \in M)$. In terms of a local coordinate system (x^μ), $\mu = 0, 1, 2, 3$, in a coordinate neighbourhood $U \subset M$, the rule (4.4) may be written as

$$A_\mu \mapsto A'_\mu = (L^{-1})A_\mu L + (L^{-1})\frac{\partial}{\partial x^\mu}L,$$

where

$$A|U = A_\mu \otimes dx^\mu. \tag{4.6}$$

REMARK 4.1. The tensor product in (4.6) means that we identify $L(T_pM, \mathfrak{u}(4, \Delta))$ with $\mathfrak{u}(4, \Delta) \otimes T_p^*M$.

Construction of the Lagrangian for this model requires the metric g to be invariant under the action of the structure group U(4, Δ). Combining this invariance with the fact that g is the primary dynamical field, we may say that the structure group U(4, Δ) acts trivially on g, i.e. g remains unchanged under the action (4.5), $g \mapsto g' = g$. It raises some objections against the status of g.

A reasonable solution (proposed by Sławianowski) consists in introduction of a field of complex linear frames $\boldsymbol{E} = (E_A)$, $A = 0, 1, 2, 3$, on M into the model as a new primary dynamical field. Then one defines g to be a byproduct of $\boldsymbol{E}$. Namely, using a field of complex linear co-frames $\widetilde{\boldsymbol{E}} = (E^A)$, $\mathbb{C}$-dual to $\boldsymbol{E}$, one builds a Hermitian tensor field

$$h[\boldsymbol{E}] = \Delta_{\overline{A}B}\overline{E^A} \otimes E^B. \tag{4.7}$$

In terms of a local coordinate system (x^μ), $\mu = 0, 1, 2, 3$, in a coordinate neighbourhood $U \subset M$,

$$\begin{aligned} h[\boldsymbol{E}]|U &= h_{\mu\nu}dx^\mu \otimes dx^\nu, \\ h_{\mu\nu} &= \Delta_{\overline{A}B}\overline{E^A{}_\mu}E^B{}_\nu. \end{aligned} \tag{4.8}$$

Then one defines a pseudo-Riemannian metric $g[\boldsymbol{E}]$ on M by

$$\begin{aligned} g[\boldsymbol{E}]|U &= g_{\mu\nu}dx^\mu \otimes dx^\nu, \\ g_{\mu\nu} &= \mathrm{Re}(h_{\mu\nu}). \end{aligned} \tag{4.9}$$

From (4.7) and (4.9) it follows that

$$h[\boldsymbol{E}L] = h[\boldsymbol{E}] \quad \text{and} \quad g[\boldsymbol{E}L] = g[\boldsymbol{E}] \tag{4.10}$$

for each mapping of class C^∞

$$L = [L^A{}_B] : M \to \mathrm{U}(4, \Delta),$$

where

$$\boldsymbol{E}L = ((\boldsymbol{E}L)_B) = (L^A{}_B E_A).$$

This means that $h[\boldsymbol{E}]$ and $g[\boldsymbol{E}]$ are invariant under the local action of the structure group $\mathrm{U}(4, \Delta)$.

The neutral signature $(++--)$ of $h[\boldsymbol{E}]$ is a preestablished parameter introduced by the matrix $[\Delta_{\overline{A}B}]$. It resembles Einstein theory formulated in tetrad terms, where the space-time metric g is a byproduct of a real tetrad field $\boldsymbol{E} = (E_A)$, $A = 0, 1, 2, 3$, on a space-time manifold, strictly speaking a real co-tetrad field $\tilde{\boldsymbol{E}} = (E^A)$ dual to $\boldsymbol{E}$,

$$g = \eta_{AB} E^A \otimes E^B.$$

The normal hyperbolic signature $(+---)$ of g is an absolute element introduced by Minkowski matrix

$$[\eta_{AB}] = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}.$$

As we mentioned in Section 3, for the models of the real tetrad field $\boldsymbol{E}$ with the Lagrangian of the Born–Infeld-type (Eqs. (3.17), (3.18) and (3.19)), for all solutions to field equations which were found by Sławianowski and the present author, $\gamma[\boldsymbol{E}]$ (Eqs. (3.17)) becomes the pseudo-Riemannian metric on M with the normal-hyperbolic signature: $(+---)$ or $(+++-)$. Obviously, this signature is not a preestablished parameter, but it is an intrinsic feature of the solutions.

Let us go back to the case when the tetrad field $\boldsymbol{E}$ is complex. According to Theorem 3.1, for the model of the complex $\boldsymbol{E}$ with Lagrangian of the Born–Infeld-type (Eqs. (2.2), (2.5), (2.6) and (2.14)), investigated in our present paper, there exist solutions to the field equations leading to the neutral signature $(++--)$ of $\gamma[\boldsymbol{E}]$ (Eqs. (2.5) and (2.6)). It allows us to expect that Sławianowski's idea for the introduction of the metric $g[\boldsymbol{E}]$ by Eqs. (4.7) and (4.9) may define a proper status of $g[\boldsymbol{E}]$ in the $\mathrm{U}(4, \Delta)$-ruled (U(2,2)-ruled) model of the bispinor field interacting with the gravitational field.

Acknowledgements

I am indebted to Professor Mayeul Arminjon for his valuable criticism and essential remarks concerning the mutual interaction of the tetrad field with the bosonic matter.

This paper contains results obtained within the framework of the research project N N501 049540 financed by Scientific Research Support Fund in 2011-2014.

REFERENCES

[1] H. H. v. Borzeszkowski and H. J. Treder: *Foundations of Physics* **28** (1998), 273.
[2] H. H. v. Borzeszkowski and H. J. Treder: Implications of Mach's Principle: Dark Matter and Observable Gravitons, in: *Causality and Locality in Modern Physics* (ed. G. Hunter et al.), p. 155, Kluwer Academic Publishers, Dordrecht Boston London 1998.
[3] D. Chruściński: *J. Phys. A: Math. Gen.* **31** (1998), 269.
[4] D. Chruściński: *Lett. Math. Phys.* **48** (1999), 385.
[5] D. Chruściński: *Rep. Math. Phys.* **45** (2000), 121.
[6] F. Finster: The Principle of the Fermionic Projector, Chapters 0-4, arXiv:hep-th/0001048 (2009).
[7] A. T. Fomenko: *Variational Methods in Topology* (in Russian), Nauka, Moscow 1982.
[8] P. Godlewski: *Rep. Math. Phys.* **35** (1995), 77.
[9] P. Godlewski: *Rep. Math. Phys.* **38** (1996), 29.
[10] P. Godlewski: *Rep. Math. Phys.* **40** (1997), 71.
[11] P. Godlewski: *Rep. Math. Phys.* **52** (2003), 81.
[12] P. Godlewski:*Demonstratio Mathematica* **38** (2005), 819.
[13] P. Godlewski: *Rep. Math. Phys.* **62** (2008), 167.
[14] P. Godlewski: *Rep. Math. Phys.* **66** (2010), 117.
[15] F. W. Hehl, J. Nitsch and P. van der Heyde: Gravitation and the Poincaré gauge field theory with quadratic Lagrangians, in: *General Relativity and Gravitation. One Hundred Years after the Birth of Albert Einstein* (ed. A. Held), Vol. 1, Chapter 11, p. 329, Plenum Press, New York 1980.
[16] H. Kleinert: *Phys. Lett. B* **174** (1986), 335.
[17] H. Kleinert: *Multivalued Fields in Condensed Matter, Electrodynamics and Gravitation*, World Scientific, Singapore 2008 (online).
[18] W. Kopczyński: *J. Phys. A: Math. Gen.* **15** (1982), 493.
[19] C. K. Möller: *Matematisk-fysiske Meddelelser udgivet af Det Kongelige Danske Videnskabernes Selskab* **39** (1978), 13.
[20] C. Pellegrini, J. Plebański: *Matematisk-fysiske Skrifter udgivet af Det Kongelige Danske Videnskabernes Selskab* **2** (1963), no. 4.
[21] J. Rzewuski: *Field Theory, Part I – Classical Theory*, PWN-Polish Scientific Publishers, Warszawa 1964.
[22] J. J. Sławianowski: *Rep. Math. Phys.* **22** (1985), 323.
[23] J. J. Sławianowski: *Rep. Math. Phys.* **23** (1986), 177.
[24] J. J. Sławianowski: *Nuovo Cim.* **106B** (1991), 645.
[25] J. J. Sławianowski: *Archives of Mechanics* **46** (1994), 375.
[26] J. J. Sławianowski: *Fortschr. Phys.* **44** (1996), 105.
[27] J. J. Sławianowski: *Rep. Math. Phys.* **38** (1996), 375.
[28] J. J. Sławianowski: *J. Tech. Phys.* **38** (1997), 3.
[29] J. J. Sławianowski: *Int. J. Theor. Phys.* **37** (1998), 411.