



# The left tail of renewal measure

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## ABSTRACT

In the paper, we find exact asymptotics of the left tail of renewal measure for a broad class of two-sided random walks. We only require that an exponential moment of the left tail is finite. Through a simple change of measure approach, our result turns out to be almost equivalent to Blackwell's Theorem.

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## 1. Introduction

Let  $(X_k)_{k \geq 1}$  be a sequence of independent copies of a random variable  $X$  with  $\mathbb{E}X > 0$  (we allow  $\mathbb{E}X = \infty$ ). Further, define  $S_n = X_1 + \dots + X_n$ ,  $n \geq 1$  and  $S_0 = 0$ . The measure defined by

$$H(B) := \sum_{n=0}^{\infty} \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R})$$

is called the *renewal measure* of  $(S_n)_{n \geq 1}$ .

We say that the distribution of a random variable  $X$  is *d-arithmetic* ( $d > 0$ ) if it is concentrated on  $d\mathbb{Z}$  and not concentrated on  $d'\mathbb{Z}$  for any  $d' > d$ . A distribution is said to be *non-arithmetic* if it is not *d-arithmetic* for any  $d > 0$ .

A fundamental result of renewal theory is the Blackwell Theorem (Blackwell, 1953): if the distribution of  $X$  is non-arithmetic, then for any  $h > 0$ ,

$$H((x, x + h]) \rightarrow \frac{h}{\mathbb{E}X} \quad \text{as } x \rightarrow \infty. \tag{1}$$

If the distribution of  $X$  is *d-arithmetic*, then for any  $h > 0$ ,

$$H((dn, dn + h]) \rightarrow \frac{d \lfloor h/d \rfloor}{\mathbb{E}X} \quad \text{as } n \rightarrow \infty. \tag{2}$$

The above results remain true if  $\mathbb{E}X = \infty$  with the usual convention that  $c/\infty = 0$  for any finite  $c$ . In the infinite-mean case the exact asymptotics of  $H((x, x + h])$  are also known. Assume that  $X$  is a non-negative random variable with a non-arithmetic law such that  $\mathbb{P}(X > x) = L(x)x^{-\alpha}$  with  $\alpha \in (0, 1)$ , where  $L$  is a slowly varying function. Then  $\mathbb{E}X = \infty$ . If  $\alpha \in (1/2, 1)$ , then without additional assumptions the so called Strong Renewal Theorem holds, for  $h > 0$ ,

$$m(x)H((x, x + h]) \rightarrow \frac{h}{\Gamma(\alpha)\Gamma(2 - \alpha)} \quad \text{as } x \rightarrow \infty, \tag{3}$$

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where  $m(x) = \int_0^x \mathbb{P}(X > t)dt \sim L(x)x^{1-\alpha}/(1 - \alpha) \rightarrow \infty$ . Here and later on  $f(x) \sim g(x)$  means that  $f(x)/g(x) \rightarrow 1$  as  $x \rightarrow \infty$ .

The case of  $\alpha \in (0, 1/2]$  is much harder and was completely solved just recently by [Caravenna and Doney \(2016\)](#). It was shown that if  $\alpha \in (0, 1/2]$  and  $X$  is a non-negative random variable with regularly varying tail, then (3) holds if and only if ([Caravenna and Doney, 2016 Proposition 1.11](#))

$$\lim_{\delta \rightarrow 0} \limsup_{x \rightarrow \infty} \int_1^{\delta x} \frac{F(x) - F(x - z)}{\bar{F}(z)z^2} dz = 0, \tag{4}$$

where  $F$  is the cumulative distribution function of  $X$  and  $\bar{F} = 1 - F$ . It was already observed by [Kevei \(2016, Theorem 3.1\)](#) that this result generalizes to  $X$  attaining negative values as well if additionally

$$\mathbb{P}(X \leq -x) = o(e^{-rx}) \quad \text{as } x \rightarrow \infty \tag{5}$$

for some  $r > 0$ . This will be our setup. Full picture of SRT for random walks is also known ([Caravenna and Doney, 2016 Theorem 1.12](#)).

It is clear that  $\lim_{x \rightarrow \infty} H((-\infty, -x)) = 0$ . There are considerably fewer papers dedicated to analysis of exact asymptotics of such object than of  $H((x, x + h])$  as in Blackwell’s Theorem. Under some additional assumptions we know more about the asymptotic behaviour of the left tail. [Stone \(1965\)](#) proved that if for some  $r > 0$  (5) holds, then for some  $r_1 > 0$ ,

$$H((-\infty, -x)) = o(e^{-r_1x}) \quad \text{as } x \rightarrow \infty. \tag{6}$$

Stone’s result was strengthened by [van der Genugten \(1969\)](#), where exact asymptotics as well the speed of convergence of the remainder term are given for  $d$ -arithmetic and spread-out laws (i.e. laws, whose  $n$ th convolution has a nontrivial absolutely continuous part for some  $n \in \mathbb{N}$ ). An important contribution regarding the asymptotics of the left tail of renewal measure was made by [Carlsson \(1983\)](#), who concerned with the case when  $\mathbb{E}|X|^m < \infty$  for some  $m \geq 2$ , but this does not fit well into our setup. We allow  $\mathbb{E}X_+ = \infty$ , but on the other hand we require that some exponential moments of  $X_-$  exist. The results mentioned above were obtained using some analytical methods, whereas we will use a simple probabilistic argument, which boils down the asymptotics of  $H((-\infty, -x))$  to the asymptotics of  $\tilde{H}((x, x + h])$ , where  $\tilde{H}$  is some new (possibly defective, see below) renewal measure.

### 1.1. Defective renewal measure

For  $\rho \in (0, 1)$  consider

$$H_\rho(B) := \sum_{n=0}^{\infty} \rho^n \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R}),$$

where  $(S_n)_{n \geq 1}$  is, as in the previous section, a random walk starting from 0.  $H_\rho$  is called a *defective renewal measure* of  $(S_n)_{n \geq 1}$ . In contrast to the renewal measure,  $H_\rho$  is a finite measure. Let  $\tau$  be independent of  $(S_n)_{n \geq 1}$  and  $\mathbb{P}(\tau = n) = (1 - \rho)\rho^n$ ,  $n = 0, 1, \dots$ . Then  $H_\rho(B) = \mathbb{P}(S_\tau \in B)/(1 - \rho)$ . It is well known that if the distribution of  $S_1$  is subexponential, then  $\mathbb{P}(S_\tau > x) \sim \mathbb{E}\tau \mathbb{P}(S_1 > x)$ . Here, we are interested in exact asymptotics of  $H_\rho(B)$  when  $B = (x, x + T]$  for any  $T > 0$ . In this context, local subexponentiality is the key concept ([Asmussen et al., 2003](#)).

Let  $\mu$  be a probability measure on  $\mathbb{R}$ . For  $T > 0$  we write  $\Delta = (0, T]$  and  $x + \Delta = (x, x + T]$ . We say that  $\mu$  belongs to the class  $\mathcal{L}_\Delta$  if  $\mu(x + \Delta) > 0$  for sufficiently large  $x$  and

$$\frac{\mu(x + s + \Delta)}{\mu(x + \Delta)} \rightarrow 1 \quad \text{as } x \rightarrow \infty, \tag{7}$$

uniformly in  $s \in [0, 1]$ .

We say that  $\mu$  is  $\Delta$ -subexponential if  $F \in \mathcal{L}_\Delta$  and

$$\mu^{*2}(x + \Delta) \sim 2\mu(x + \Delta).$$

Then we write  $\mu \in \mathcal{S}_\Delta$ . Finally,  $\mu$  is called *locally subexponential* if  $\mu \in \mathcal{S}_\Delta$  for any  $T > 0$ . We denote this class by  $\mathcal{S}_{loc}$ .

The following theorem is an obvious conclusion from [Watanabe and Yamamuro \(2009, Theorem 1.1\)](#).

**Theorem 1.1.** Assume that  $\mu$  is a probability measure on  $\mathbb{R}$  such that

$$\int_{\mathbb{R}} e^{-\varepsilon x} \mu(dx) < \infty \quad \text{for some } \varepsilon > 0.$$

For  $0 < \rho < 1$  define

$$\eta = \sum_{n=0}^{\infty} \rho^n \mu^{*n}.$$

Then  $\mu \in S_\Delta$  if and only if  $\eta/(1 - \rho) \in S_\Delta$  if and only if

$$\eta(x + \Delta) \sim \frac{\rho}{(1 - \rho)^2} \mu(x + \Delta).$$

Some examples of measures from  $S_{loc}$  may be found in [Asmussen et al. \(2003, Section 4\)](#).

**2. Main result**

Assume that  $X$  is a random variable with  $\mathbb{E}X \in (0, \infty]$ . We define the Laplace transform of the distribution of  $X$  by

$$g(\theta) := \mathbb{E}e^{-\theta X}.$$

Function  $g$  is convex and lower-semicontinuous. We are interested in a situation of an exponentially decaying left tail, that is,

$$\mathbb{P}(X < 0) > 0 \quad \text{and} \quad g(\theta) < \infty \quad \text{for some } \theta > 0. \tag{8}$$

Under (8) we define

$$\kappa := \sup\{\theta > 0 : g(\theta) < 1\} \quad \text{and} \quad \rho := g(\kappa). \tag{9}$$

Since  $g'(0) = -\mathbb{E}X < 0$ ,  $\kappa$  is strictly positive. Moreover, we have  $g(\theta) \rightarrow \infty$  as  $\theta \rightarrow \infty$  and thus  $\kappa$  is finite. In general we have  $0 < \rho \leq 1$  and a sufficient condition for  $\rho = 1$  is that  $g(\theta) < \infty$  for all  $\theta > 0$ .

**Theorem 2.1.** Assume  $X$  is a random variable with a positive (possibly infinite) expectation such that (8) holds. Let  $H$  be the renewal measure of  $(S_n)_{n \geq 0}$ , where  $S_n = \sum_{k=1}^n X_k$  for  $n \in \mathbb{N}$ ,  $S_0 = 0$  and  $X_k$  are independent copies of  $X$ . Define  $\kappa$  and  $\rho$  as in (9).

(a) Assume that  $\rho = 1$ .

(a-i) Assume that  $X$  has a non-arithmetic distribution. Then

$$\lim_{x \rightarrow \infty} e^{\kappa x} H((-\infty, -x)) = \frac{1}{\kappa g'(\kappa)} \in [0, \infty).$$

Moreover, if

$$\mathbb{E}e^{-\kappa X} \mathbf{1}_{\{-X > t\}} \sim \frac{L(t)}{t^\alpha} \tag{10}$$

for some  $\alpha \in (0, 1)$  and a slowly varying function  $L$ , then  $g'(\kappa) = \infty$ . For  $\alpha \in (0, 1/2]$ , assume additionally that  $F(t) = \mathbb{E}e^{-\kappa X} \mathbf{1}_{\{-X \leq t\}}$  satisfies (4). In such case,

$$e^{\kappa x} H((-\infty, -x)) \sim \frac{1}{\Gamma(\alpha)\Gamma(2 - \alpha)} \frac{1}{\kappa m(x)},$$

where  $m(x) \sim L(x)x^{1-\alpha}/(1 - \alpha)$ .

(a-ii) Assume that  $X$  has a  $d$ -arithmetic distribution. Then

$$\lim_{n \rightarrow \infty} e^{\kappa dn} H((-\infty, -nd)) = \frac{d}{(e^{\kappa d} - 1)g'(\kappa)}.$$

(b) If  $\rho < 1$ , then

$$\lim_{x \rightarrow \infty} e^{\kappa x} H((-\infty, -x)) = 0. \tag{11}$$

Moreover, if  $\rho^{-1} \mathbb{E}e^{-\kappa X} \mathbf{1}_{\{-X \in \cdot\}} \in S_{loc}$ , then

$$e^{\kappa x} H((-\infty, -x)) \sim \frac{\mathbb{E}e^{-\kappa X} \mathbf{1}_{\{x < -X \leq x+1\}}}{\kappa(1 - \rho)^2}.$$

**Remark 2.2.** Condition (10) is implied by

$$\mathbb{P}(-X > t) = \frac{\alpha}{\kappa} \frac{L(t)}{t^{\alpha+1}} e^{-\kappa t}, \quad t > 0.$$

Indeed, for any slowly varying function  $L$  and  $\beta < -1$ , [Bingham et al. \(1989, Proposition 1.5.10\)](#) asserts that

$$\int_x^\infty t^\beta L(t) dt \sim x^{\beta+1} L(x) / (-\beta - 1).$$

**Remark 2.3.** Under the same assumptions, a stronger result concerning (a-ii) is proved in [van der Genugten \(1969, Theorem 2\)](#) (the remainder term is also exponential).

**Remark 2.4.** If  $X$  has a non-arithmetic distribution, for any  $\delta > 0$ , we obtain “more local” behaviour:

$$\lim_{x \rightarrow \infty} e^{\kappa x} H((-\infty, -x - \delta, -x)) = \frac{1 - e^{-\delta \kappa}}{\kappa g'(\kappa)}.$$

**Proof of Theorem 2.1.** Note that  $g'(\kappa) = -\mathbb{E}X e^{-\theta X}$  is positive ( $1 = g(0) = g(\kappa)$  and  $g$  is convex), but may be infinite.

Define  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$  and let  $\mathcal{F}_\infty$  be the smallest  $\sigma$ -field containing all  $\mathcal{F}_n$ . On  $(\Omega, \mathcal{F}_\infty)$  we define a new measure  $\mathbb{Q}$  via projections

$$\mathbb{Q}((X_1, \dots, X_n) \in B) = \rho^{-n} \mathbb{E} e^{-\kappa S_n} \mathbf{1}_{\{(X_1, \dots, X_n) \in B\}}, \quad B \in \mathcal{B}(\mathbb{R}^n),$$

where  $S_n = X_1 + \dots + X_n$ ,  $n \in \mathbb{N}$  and  $S_0 = 0$ . By the definition of  $\rho$ ,  $\mathbb{Q}$  is a probability measure. Moreover,  $(X_n)_{n \geq 1}$  is an iid sequence under  $\mathbb{Q}$  as well. Let  $\mathbb{E}_\mathbb{Q}$  denote the corresponding expectation. For any Borel function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  one has

$$\mathbb{E}f(S_n) = \rho^n \mathbb{E}_\mathbb{Q} e^{-\kappa S_n} f(-S_n).$$

Thus,

$$\mathbb{P}(S_n < -x) = \rho^n \mathbb{E}_\mathbb{Q} e^{-\kappa S_n} \mathbf{1}_{S_n > x} = \rho^n \int_{(x, \infty)} e^{-\kappa t} \mathbb{Q}_X^{*n}(dt).$$

Moreover, observe that  $\mathbb{E}_\mathbb{Q} X = -\mathbb{E}X e^{-\kappa X} = g'(\kappa) \in (0, \infty]$ , thus  $(S_n)_n$  has a positive drift under  $\mathbb{Q}$  as well. Hence, for  $x > 0$ ,

$$H((-\infty, -x)) = H_\mathbb{P}((-\infty, -x)) = \sum_{n=1}^\infty \mathbb{P}(S_n < -x) = \int_{(x, \infty)} e^{-\kappa t} H_\mathbb{Q}(dt),$$

where  $H_\mathbb{Q} = \sum_{n=0}^\infty \rho^n \mathbb{Q}_X^{*n}$  is the (defective if  $\rho < 1$ ) renewal measure of  $(S_n)_{n \geq 0}$  under  $\mathbb{Q}$ .

Writing  $e^{-\kappa t} = \kappa \int_t^\infty e^{-\kappa s} ds$ , through Tonelli’s Theorem, we arrive at key identity:

$$H_\mathbb{P}((-\infty, -x)) = \kappa \int_x^\infty e^{-\kappa s} H_\mathbb{Q}((x, s]) ds = \kappa e^{-\kappa x} \int_0^\infty e^{-\kappa h} H_\mathbb{Q}((x, x + h]) dh. \tag{12}$$

Consider first the case of  $\rho = 1$ . For any renewal measure  $H$  we have  $H((x, x + h]) \leq \alpha h + \beta$  for some  $\alpha, \beta > 0$  and all  $x$ , thus by Lebesgue’s Dominated Convergence Theorem and (1),

$$\lim_{x \rightarrow \infty} e^{\kappa x} H_\mathbb{P}((-\infty, -x)) = \kappa \int_0^\infty e^{-\kappa h} \lim_{x \rightarrow \infty} H_\mathbb{Q}((x, x + h]) dh = \frac{1}{\kappa \mathbb{E}_\mathbb{Q} X},$$

which gives the first part of (a-i). For (a-ii) use (2), instead of (1).

For the second part of (a-i), observe that

$$\mathbb{Q}(X > t) = \mathbb{E} e^{-\kappa X} \mathbf{1}_{\{-X > t\}} = \frac{L(t)}{t^\alpha},$$

thus the result follows by the Strong Renewal Theorem.

If  $\rho < 1$ , then  $H_\mathbb{Q}$  is a finite measure and (11) follows again by Lebesgue’s Dominated Convergence Theorem.

Consider now the case, when  $\mathbb{Q}_X \in \mathcal{S}_{loc}$ . Since  $\mathbb{E}_\mathbb{Q} e^{-\kappa X} = 1 < \infty$ , by [Theorem 1.1](#), we have  $(1 - \rho)H_\mathbb{Q} \in \mathcal{S}_{loc}$  and

$$H_\mathbb{Q}((x, x + 1]) \sim \frac{\rho}{(1 - \rho)^2} \mathbb{Q}_X((x, x + 1]). \tag{13}$$

Define  $L(y) := H_\mathbb{Q}((\log y, \log y + 1])$ . By (7),  $L$  is a slowly varying function. Moreover, for any  $h > 0$

$$\frac{H_\mathbb{Q}((x, x + h])}{H_\mathbb{Q}((x, x + 1])} \leq \sum_{n=1}^{[h]} \frac{H_\mathbb{Q}((x + n - 1, x + n])}{H_\mathbb{Q}((x, x + 1])} = \sum_{n=1}^{[h]} \frac{L(e^{x+n-1})}{L(e^x)}.$$

By Potter bounds ([Bingham et al., 1989 Theorem 1.5.6](#)) for any  $\varepsilon > 0$  and  $C > 1$  there exists  $x_0$  such that for  $x > x_0$ ,

$$\frac{H_\mathbb{Q}((x, x + h])}{H_\mathbb{Q}((x, x + 1])} \leq \sum_{n=1}^{[h]} C e^{\varepsilon(n-1)} \leq C [h] e^{\varepsilon [h]}.$$

Observe that (7) implies for  $h > 0$ ,

$$H_\mathbb{Q}((x, x + h]) \sim h H_\mathbb{Q}((x, x + 1]).$$

Indeed, for  $h = k/n \in \mathbb{Q}_+$  one gets

$$H_{\mathbb{Q}}\left(\left(x, x + \frac{k}{n}\right]\right) = \sum_{i=1}^k H_{\mathbb{Q}}\left(\left(x + \frac{i-1}{n}, x + \frac{i}{n}\right]\right) \sim k H_{\mathbb{Q}}\left(\left(x, x + \frac{1}{n}\right]\right)$$

and

$$H_{\mathbb{Q}}\left(\left(x, x + \frac{1}{n}\right]\right) \sim \frac{1}{n} \sum_{i=1}^n H_{\mathbb{Q}}\left(\left(x + \frac{i-1}{n}, x + \frac{i}{n}\right]\right) = \frac{1}{n} H_{\mathbb{Q}}\left(\left(x, x + 1\right]\right).$$

By monotonicity,  $f(h) := \lim_{x \rightarrow \infty} H_{\mathbb{Q}}\left(\left(x, x + h\right]\right) / H_{\mathbb{Q}}\left(\left(x, x + 1\right]\right)$  exists for all  $h > 0$  and  $f(h) = h$ . Thus, by (12) and Lebesgue's Dominated Convergence Theorem we conclude that

$$\lim_{x \rightarrow \infty} e^{\kappa x} \frac{H_{\mathbb{P}}\left(\left(-\infty, -x\right)\right)}{H_{\mathbb{Q}}\left(\left(x, x + 1\right]\right)} = \kappa \int_0^{\infty} e^{-\kappa h} \lim_{x \rightarrow \infty} \frac{H_{\mathbb{Q}}\left(\left(x, x + h\right]\right)}{H_{\mathbb{Q}}\left(\left(x, x + 1\right]\right)} dh = 1/\kappa.$$

The use of (13) completes the proof.  $\square$

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