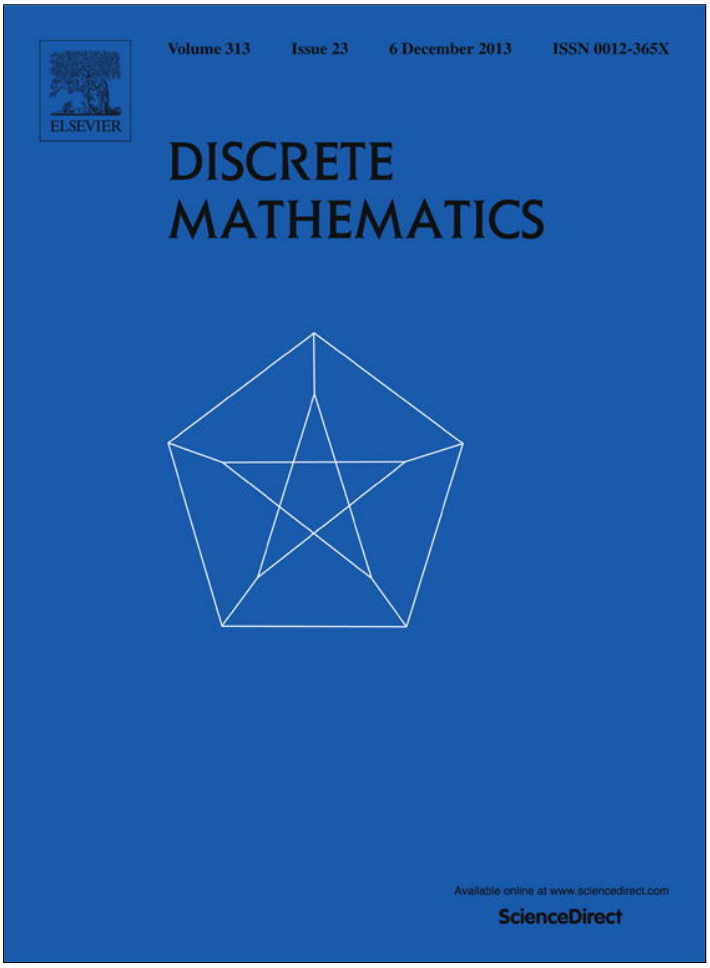
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Minimum size transversals in uniform hypergraphs

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ABSTRACT

A set of vertices in a hypergraph which meets all the edges is called a transversal. The transversal number $\tau(H)$ of a hypergraph H is the minimum cardinality of a transversal in H. A classical greedy algorithm for constructing a transversal of small size selects in each step a vertex which has the largest degree in the hypergraph formed by the edges not met yet. The analysis of this algorithm (by Chvátal and McDiarmid (1992) [3]) gave some upper bounds for $\tau(H)$ in a uniform hypergraph H with a given number of vertices and edges. We discuss a variation of this greedy algorithm. Analyzing this new algorithm, we obtain upper bounds for $\tau(H)$ which improve the bounds by Chvátal and McDiarmid.

1. Introduction

By a c-uniform hypergraph we mean a pair $H = (X, \mathcal{E})$, where X is a finite set and $\mathcal{E}$ is a set of pairwise distinct c-element subsets of X. The elements of X are called the vertices of H and the elements of $\mathcal{E}$ the edges of H. Let x be a vertex in a hypergraph $H = (X, \mathcal{E})$. We define $\mathcal{E}_x = \{E \in \mathcal{E} : x \in E\}$. By the *degree* of x in H (denoted by $\deg_{\mathcal{E}} x$) we mean $|\mathcal{E}_x|$. We denote by $\Delta(\mathcal{E})$ or $\Delta(H)$ the maximum degree of H. A uniform hypergraph is r-regular if all degrees of its vertices are equal to r.

By a *transversal* in a c-uniform hypergraph we mean a set of vertices which meets every edge. The *transversal number* $\tau(H)$ of a hypergraph H is the minimum cardinality of a transversal in H. In this paper we address the question on the best upper bound for $\tau(H)$ in a c-uniform hypergraph H with n vertices and m edges.

Let, for $1 \le c \le n$ and $m \le \binom{n}{c}$, $g(c, m, n)$ be the smallest integer such that every c-uniform hypergraph H with n vertices and m edges has $\tau(H) \le g(c, m, n)$.

Obviously, $g(1, m, n) = m$. For $c = 2$, i.e. for graphs, the values of $g(2, m, n)$ can be deduced from the Turán theorem [8] (see [3]):

$$g(2, m, n) = \left\lfloor \frac{2}{(d+1)d} m + \frac{d-1}{d+1} n \right\rfloor,$$

for $(d-1)\frac{n}{2} < m \le d\frac{n}{2}$ and $d = 1, 2, \ldots, n-1$.

For $c \ge 3$ much less is known. Clearly, $g(c, m, n) = m$, for $m \le \frac{n}{c}$.

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Tuza [9] showed that $g(3, m, n) \leq \frac{1}{4}m + \frac{1}{4}n$. This result was generalized by Chvátal and McDiarmid [3] who proved that

$$g(c, m, n) \leq \frac{\lfloor c/2 \rfloor m + n}{\lfloor 3c/2 \rfloor}. \tag{1}$$

It was shown in [3] that this inequality is the best possible linear upper bound for $g(c, m, n)$ when $\frac{n}{c} \leq m \leq \frac{n}{\lfloor c/2 \rfloor}$. More precisely, for every bound $g(c, m, n) \leq am + bn$ (where a and b are real numbers that depend on c only), the inequality $\frac{\lfloor c/2 \rfloor m + n}{\lfloor 3c/2 \rfloor} \leq am + bn$ holds for $\frac{n}{c} \leq m \leq \frac{n}{\lfloor c/2 \rfloor}$.

In the same paper the authors also proved that $g(3, m, n) \leq \frac{1}{6}m + \frac{1}{3}n$. This bound is the best possible for $n \leq m \leq \frac{4n}{3}$ (in the sense defined in the preceding paragraph). These results occur also implicitly in Sidorenko [6].

Lai and Chang [5] found the bound $g(4, m, n) \leq \frac{2}{9}(m + n)$. This result was improved by Thomassé and A. Yeo [7] who showed the inequality $g(4, m, n) \leq \frac{4m+5n}{21}$. This upper bound for $g(4, m, n)$ is better than the bound proved in [5] for all $m > \frac{n}{2}$ and, in fact, it is the best possible linear upper bound for $g(4, m, n)$ when $\frac{n}{2} \leq m \leq n$.

A related result was proved by Alon [1] who showed that the smallest constant γ_c such that $g(c, m, n) \leq \gamma_c(m + n)$ satisfies the equality $\gamma_c = (1 + o(1))\frac{\log c}{c}$.

General linear upper bounds for $g(c, m, n)$ were found by Chvátal and McDiarmid [3]. They turn out to be upper bounds for a transversal found by the following well-known greedy algorithm.

Algorithm Greedy(X, $\mathcal{E}$)

$\mathcal{A} \leftarrow \mathcal{E}$
$T \leftarrow \emptyset$
while $\mathcal{A} \neq \emptyset$
  do select a vertex $x \in X$ that maximizes $\deg_{\mathcal{A}} x = |\mathcal{A}_x|$
  $\mathcal{A} \leftarrow \mathcal{A} - \mathcal{A}_x$
  $T \leftarrow T \cup \{x\}$
return T

Chvátal and McDiarmid [3] proved that for a c-uniform hypergraph with n vertices and m edges this algorithm produces a transversal T whose size satisfies the following inequalities

$$|T| \leq m\frac{(k-1)!(c-1)^{k-1}}{\prod_{j=2}^{k}(j(c-1)+1)} + n\left[1 - \frac{k!(c-1)^{k-1}}{\prod_{j=2}^{k}(j(c-1)+1)}\right], \tag{2}$$

for every $k = 1, 2, \ldots$. For $c = 3$, the inequalities reduce to

$$|T| \leq m\frac{3}{2k}\frac{4^k(k!)^2}{(2k+1)!} + n\left[1 - \frac{3}{2}\frac{4^k(k!)^2}{(2k+1)!}\right]. \tag{3}$$

The upper bounds for $|T|$ in (2) and (3) are tight in the sense that there are c-uniform hypergraphs with n vertices and m edges for which the greedy algorithm produces transversals of the sizes equal to the expressions on the right hand sides of the inequalities (2) and (3). Clearly, these expressions are also upper bounds for $g(c, m, n)$. However, they are not the best linear upper bounds for $g(c, m, n)$ in any range of m except the trivial bound $g(c, m, n) \leq m$ (the case of $k = 1$ in (2) and (3)) which is the best for $m \leq \frac{n}{c}$.

In this paper we consider a certain modification of the algorithm Greedy which also produces transversals in uniform hypergraphs. We call this modified algorithm SuperGreedy. We show that this algorithm produces transversals T of sizes for which we are able to prove upper bounds that are smaller than the bounds (2) and (3). This way we get better linear upper bounds for $g(c, m, n)$ but we are not able to prove if any of these bounds is the best linear upper bound for some range of m (except bounds which have already been known to be the best).

To formulate our main result, we need to introduce a certain sequence.

Let, for $c \geq 4$, p_j be a sequence defined recursively as follows

$$p_j = \frac{(j(c-1) - 2c)p_{j-1} + p_{j-2}}{j(c-1)}, \tag{4}$$

for $j = 5, 6, \ldots$,

$$p_1 = \frac{1}{c}, p_2 = -\frac{1}{\lfloor 3c/2 \rfloor}, \tag{5}$$

$$p_3 = \begin{cases} -\dfrac{2}{9c(c-1)} & \text{for } c \text{ even} \\ 0 & \text{for } c \text{ odd} \end{cases} \tag{6}$$

and

$$p_4 = \begin{cases} -\dfrac{(c-2)(3c+1)}{36c(c-1)^2} & \text{for } c \text{ even} \\ -\dfrac{1}{4(3c-1)} & \text{for } c \text{ odd.} \end{cases} \tag{7}$$

For $c = 3$, we define p_j as follows

$$p_j = \frac{(2j-6)p_{j-1} + p_{j-2}}{2j}, \tag{8}$$

for $j = 7, 8, \ldots,$

$$p_1 = \frac{1}{3}, \qquad p_2 = -\frac{1}{4}, \qquad p_3 = 0, \qquad p_4 = -\frac{1}{36}, \qquad p_5 = -\frac{1}{60} \quad \text{and} \quad p_6 = -\frac{1}{120}. \tag{9}$$

Here is our main result.

Theorem 1.1. *If H is a c-uniform hypergraph with n vertices and m edges then*

$$\tau(H) \le \left(c \sum_{i=1}^{k} p_i \right) m - \left(\sum_{i=1}^{k} (i-1) p_i \right) n, \tag{10}$$

for every $k = 1, 2, \ldots$.

The paper is organized as follows. In Section 2 we introduce the algorithm SuperGreedy and show that the numbers of vertices added to the constructed transversal in the subsequent passes of the main loop of this algorithm satisfy some linear inequalities. In Section 3 we show that the problem of finding an upper bound for the cardinality of a transversal constructed by the algorithms SuperGreedy or Greedy can be formulated as a linear programming problem. We demonstrate how to apply this approach to give an alternative proof of the inequalities (2) and (3). Some properties of Steiner systems which are useful later are proved in Section 4. In Section 5 we strengthen the inequalities shown in Section 2 in the case of the SuperGreedy algorithm. In Section 6 we define another linear program and apply it to prove the main result of the paper. Finally, in Section 7 we compare our results to results proved by other authors and point out some connections to other combinatorial problems.

2. Greedy algorithms

A set S of vertices in a hypergraph H is said to be *strongly independent* in H if no edge in H contains two or more vertices in S.

Let us observe that in the algorithm Greedy, while constructing a transversal T, we have a choice when there are at least two vertices of the largest degree in the hypergraph H' defined by the edges not removed from the hypergraph yet. Moreover, the vertices which have the same degree when they are added to the transversal T form a strongly independent set in H'.

Therefore the algorithm Greedy can be rewritten in the following form.

Algorithm Greedy$(X, \mathcal{E})$
$\mathcal{A} \leftarrow \mathcal{E}$
$T \leftarrow \emptyset$
$d \leftarrow \Delta(\mathcal{E})$
for $k = d$ **down to** 1 **do**
  $V_k \leftarrow \{x \in X - T : \deg_{\mathcal{A}} x = k\}$
  select a maximal subset $U_k \subseteq V_k$ which is strongly independent in $\mathcal{A}$
  $\mathcal{A} \leftarrow \mathcal{A} - \bigcup_{x \in U_k} \mathcal{A}_x$
  $T \leftarrow T \cup U_k$
end of for
return T

In this paper we discuss an algorithm which can be obtained from the algorithm Greedy by the following modification. Instead of choosing, in the hypergraph defined by the edges not removed from the hypergraph yet, the vertices of the largest degree such that they form a maximal strongly independent subset of V_k, we choose them such that they form a maximum-sized strongly independent subset of V_k. In other words this new algorithm (which we call SuperGreedy) looks as follows.

Algorithm SuperGreedy$(X, \mathcal{E})$

1 $\mathcal{A} \leftarrow \mathcal{E}$
2 $T \leftarrow \emptyset$
3 $d \leftarrow \Delta(\mathcal{E})$
4 **for** $k = d$ **down to** 1 **do**
5 $\quad V_k \leftarrow \{x \in X - T : \deg_{\mathcal{A}} x = k\}$
6 $\quad$ select a maximum-sized subset $U_k \subseteq V_k$ which is strongly independent in $\mathcal{A}$
7 $\quad \mathcal{A} \leftarrow \mathcal{A} - \bigcup_{x \in U_k} \mathcal{A}_x$
8 $\quad T \leftarrow T \cup U_k$
9 **end of for**
10 **return** T

Clearly, unlike the algorithm Greedy, the SuperGreedy algorithm is not polynomial (unless $\mathbf{P} = \mathbf{NP}$) so its practical use is very limited. However, in this paper we are going to use this algorithm to prove some new upper bounds on the minimum size of a transversal in a hypergraph. From this point of view the time complexity of the algorithm SuperGreedy is not an issue.

We shall show now some inequalities that must be satisfied by the numbers of vertices $|U_k|$ adjoined to the transversal T in the passes of the "for loop" of the algorithms Greedy or SuperGreedy.

For a c-uniform hypergraph with n vertices, m edges and maximum degree equal to d, let t_k, $k = d, d-1, \ldots, 1$, be the cardinality of the set U_k in line 6 of the algorithm Greedy or SuperGreedy.

Lemma 2.1. *For $i = 2, 3, \ldots, d$,*

$$\sum_{j=i}^{d} (jc - i + 1)t_j \geq mc - n(i-1) \tag{11}$$

and

$$\sum_{j=1}^{d} jct_j = mc. \tag{12}$$

Proof. We observe that at the end of the pass of the "for loop" for $k = i$

$$\sum_{x \in X} \deg_{\mathcal{A}} x = mc - \sum_{j=i}^{d} jct_j. \tag{13}$$

On the other hand the number of vertices of positive degree in $(X, \mathcal{A})$ at the end of this pass of the "for loop" is at most $n - \sum_{j=i}^{d} t_j$ and $\Delta(\mathcal{A}) \leq i - 1$ so

$$\sum_{x \in X} \deg_{\mathcal{A}} x \leq (i-1)\left(n - \sum_{j=i}^{d} t_j\right). \tag{14}$$

By (13) and (14), for $i = 2, 3, \ldots, d$,

$$mc - \sum_{j=i}^{d} jct_j \leq (i-1)\left(n - \sum_{j=i}^{d} t_j\right). \tag{15}$$

For $i = 1$ we get a stronger condition

$$mc - \sum_{j=1}^{d} jct_j = 0, \tag{16}$$

because at the end of the pass of the "for loop" for $k = 1$, $\sum_{x \in X} \deg_{\mathcal{A}} x = 0$. The conditions (15) and (16) are equivalent to (11) and (12). □

We shall show now that the inequality (11) can be strengthened for the algorithm SuperGreedy and $i = 2$. To do it we need the next two lemmas.

Lemma 2.2. *Let H be a c-uniform hypergraph H with m edges and $\Delta(H) = 2$. Then*

$$\tau(H) = m - t,$$

where t is the maximum cardinality of a strongly independent subset of the set of vertices of degree 2.

Proof. Let D be the set of vertices of degree 2 in H and let T be a transversal in H. We denote by I a strongly independent subset of $T \cap D$ of maximum cardinality. Clearly, there are $m - 2|I|$ edges of H not intersecting I and, by maximality of I, they are pairwise disjoint. Hence $|T - I| \geq m - 2|I|$, so $|T| \geq m - |I| \geq m - t$.

On the other hand, let I_0 be a strongly independent subset of D of maximum cardinality t. There are $m - 2t$ edges of H not intersecting I_0. Thus I_0 can be extended by $m - 2t$ elements to form a transversal of cardinality $m - t$ in H. □

Lemma 2.3. *If, for a c-uniform hypergraph $H = (X, \mathcal{A})$, $\Delta(H) = 2$ then there is a set $U_2 \subseteq V_2 = \{x \in X : \deg_{\mathcal{A}} x = 2\}$, which is strongly independent in H, such that*

$$|U_2| \geq \frac{|V_2|}{\lfloor 3c/2 \rfloor}. \tag{17}$$

Proof. By Lemma 2.2 and the inequality (1), there is a strongly independent set $U_2 \subseteq V_2$ such that

$$m - |U_2| = \tau(H) \leq g(c, m, n) \leq \frac{\lfloor c/2 \rfloor m + n}{\lfloor 3c/2 \rfloor}.$$

This is obviously equivalent to (17) as $cm - n = |V_2|$. □

Lemma 2.4. *For a c-uniform hypergraph with n vertices, m edges and maximum degree equal to d let t_k, $k = d, d-1, \ldots, 1$, be the cardinality of the set U_k in line 6 of the algorithm* SuperGreedy. *Then*

$$\lfloor 3c/2 \rfloor t_2 + \sum_{j=3}^{d} (jc - 1) t_j \geq mc - n. \tag{18}$$

Proof. We observe that at start of the pass of the "for loop" for $k = 2$ the number of edges in the hypergraph $(X, \mathcal{A})$ is

$$m' = |\mathcal{A}| = \frac{1}{c} \sum_{x \in X} \deg_{\mathcal{A}} x = m - \sum_{j=3}^{d} j t_j \tag{19}$$

and the number of vertices of positive degree is

$$n' \leq n - \sum_{j=3}^{d} t_j. \tag{20}$$

Obviously, the maximum degree in this hypergraph is at most 2.

Applying Lemma 2.3 we get

$$t_2 = |U_2| \geq \frac{|V_2|}{\lfloor 3c/2 \rfloor} = \frac{cm' - n'}{\lfloor 3c/2 \rfloor}, \tag{21}$$

so by (19) and (20)

$$\lfloor 3c/2 \rfloor t_2 \geq m'c - n' \geq mc - n - \sum_{j=3}^{d} (jc - 1) t_j. \quad \square \tag{22}$$

3. A linear program

Clearly, the size of a transversal T constructed by the algorithm Greedy or SuperGreedy is equal to $|T| = t_1 + t_2 + \cdots + t_d$, where t_k, $k = 1, 2, \ldots, d$, is the cardinality of the set U_k in line 6 of the algorithm Greedy or SuperGreedy and d is the maximum degree of the input hypergraph. Therefore, an upper bound for the cardinality of the transversal T can be obtained by maximizing the sum $t_1 + t_2 + \cdots + t_d$ subject to the constraints (11), (12) and the inequalities

$$t_k \geq 0,$$

for $k = 1, 2, \ldots, d$.

This is a linear programming problem. We shall later strengthen the inequalities (11) in the case of the algorithm SuperGreedy. We need two lemmas that will be useful in solving these linear programming problems.

Lemma 3.1. *Let* $\mathbf{A}$ *be a* $d \times d$ *nonsingular upper-right triangular matrix, let* $\mathbf{1} = [1, 1, \ldots, 1]$ *be a* $1 \times d$ *vector and let* $\mathbf{e} = [e_1, e_2 \ldots, e_d] = \mathbf{1} \cdot \mathbf{A}^{-1}$. *Then*

$$e_1 = \frac{1}{a_{1,1}}, \qquad e_2 = \frac{a_{1,1} - a_{1,2}}{a_{1,1}a_{2,2}} \tag{23}$$

and

$$e_j = \frac{e_{j-1}(2a_{j-1,j-1} - a_{j-1,j}) + \sum_{i=1}^{j-2} e_i(2a_{i,j-1} - a_{i,j} - a_{i,j-2})}{a_{j,j}},$$

for $j = 3, \ldots, d$.

Proof. Obviously, $\mathbf{e} \cdot \mathbf{A} = \mathbf{1}$, so

$$\sum_{i=1}^{j} e_i a_{i,j} = 1, \quad \text{for every } j = 1, \ldots, d. \tag{24}$$

In particular $e_1 a_{1,1} = 1$ and $e_1 a_{1,2} + e_2 a_{2,2} = 1$ so $e_1 = \frac{1}{a_{1,1}}$ and $e_2 = \frac{a_{1,1}-a_{1,2}}{a_{1,1}a_{2,2}}$.

Moreover, by (24), for $j \geq 3$,

$$\begin{aligned} 0 = 1 - 2 \cdot 1 + 1 &= \sum_{i=1}^{j} e_i a_{i,j} - 2\sum_{i=1}^{j-1} e_i a_{i,j-1} + \sum_{i=1}^{j-2} e_i a_{i,j-2} \\ &= e_j a_{j,j} + e_{j-1}a_{j-1,j} - 2e_{j-1}a_{j-1,j-1} + \sum_{i=1}^{j-2} e_i(a_{i,j} - 2a_{i,j-1} + a_{i,j-2}) \end{aligned}$$

so

$$e_j = \frac{e_{j-1}(2a_{j-1,j-1} - a_{j-1,j}) + \sum_{i=1}^{j-2} e_i(2a_{i,j-1} - a_{i,j} - a_{i,j-2})}{a_{j,j}}. \quad \square$$

Lemma 3.2. *Let* $\mathbf{A}$ *be a* $d \times d$ *nonsingular upper-right triangular matrix such that*

(i) $a_{i,j} > 0$ *for* $1 \leq i \leq j \leq d$,
(ii) $a_{1,j} = a_{1,1} + c(j-1)$, *for* $1 \leq j \leq d$,
(iii) $a_{i,j} = a_{i,i+1} + c(j-i-1)$, *for* $3 \leq i+1 \leq j \leq d$,
(iv) $2a_{j-1,j-1} - a_{j-1,j} \geq 0$, *for* $3 \leq j \leq d$, *and*
(v) $2a_{j-2,j-1} - a_{j-2,j} - a_{j-2,j-2} \geq 0$, *for* $3 \leq j \leq d$.

Let $\mathbf{1} = [1, 1, \ldots, 1]$ *be a* $1 \times d$ *vector and let* $\mathbf{e} = [e_1, e_2 \ldots, e_d] = \mathbf{1} \cdot \mathbf{A}^{-1}$. *Then, for* $j = 3, \ldots, d$,

$$e_j = \frac{e_{j-1}(2a_{j-1,j-1} - a_{j-1,j}) + e_{j-2}(2a_{j-2,j-1} - a_{j-2,j} - a_{j-2,j-2})}{a_{j,j}}. \tag{25}$$

Moreover, for $j = 2, \ldots, d$,

$$e_j \leq 0. \tag{26}$$

Proof. We observe that, for $i = 2, \ldots, j-3$, (iii) implies

$$2a_{i,j-1} - a_{i,j} - a_{i,j-2} = 2(a_{i,i+1} + c(j-1-i-1)) - a_{i,i+1} - c(j-i-1) - a_{i,i+1} - c(j-2-i-1) = 0.$$

Moreover, by (ii)

$$2a_{1,j-1} - a_{1,j} - a_{1,j-2} = 2(a_{1,1} + c(j-2)) - a_{1,1} - c(j-1) - a_{1,1} - c(j-3) = 0.$$

Lemma 3.1 implies now (25).

To prove that $e_j \leq 0$, for $j = 2, \ldots, d$, we proceed by induction on j. By Lemma 3.1, (ii) and (i), $e_2 = \frac{a_{1,1}-a_{1,2}}{a_{1,1}a_{2,2}} = \frac{-c}{a_{1,1}a_{2,2}} \leq 0$. We assume that $e_i \leq 0$, for $2 \leq i \leq j-1 \leq d-1$. By (25), (iv), (v), (ii), (i) and the induction hypothesis,

$$e_j \leq 0. \quad \square$$

Theorem 3.3. *For a c-uniform hypergraph with n vertices, m edges and maximum degree equal to d, the cardinality of the transversal T constructed by the algorithm* Greedy *or* SuperGreedy *satisfies the inequality*

$$|T| \leq \mathbf{1} \cdot \mathbf{A}^{-1} \cdot \mathbf{b}, \tag{27}$$

where

$$\mathbf{A} = \begin{bmatrix} c & 2c & 3c & 4c & 5c & \cdots & dc \\ 0 & 2c-1 & 3c-1 & 4c-1 & 5c-1 & \cdots & dc-1 \\ 0 & 0 & 3c-2 & 4c-2 & 5c-2 & \cdots & dc-2 \\ 0 & 0 & 0 & 4c-3 & 5c-3 & \cdots & dc-3 \\ 0 & 0 & 0 & 0 & 5c-4 & \cdots & dc-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & dc-d+1 \end{bmatrix},$$

$\mathbf{b} = [mc, mc-n, mc-2n, \ldots, mc-(\lceil p \rceil - 2)n, mc-(\lceil p \rceil - 1)n, 0, \ldots, 0]^T$, $\mathbf{1} = [1, 1, \ldots, 1]$ *and* $p = \frac{mc}{n} \leq d$ *is the average degree of a vertex in the hypergraph.*

Proof. By the definition, the entries of $\mathbf{A}$ are $a_{i,j} = cj - i + 1$, for $1 \leq i \leq j \leq d$. We observe that, for $j = 3, \ldots, d$, $2a_{j-2,j-1} - a_{j-2,j} - a_{j-2,j-2} = 0$ and $2a_{j-1,j-1} - a_{j-1,j} = (c-1)(j-2) \geq 0$. Thus the matrix $\mathbf{A}$ satisfies the conditions (i)–(v) of Lemma 3.2.

Let $\mathbf{b} = [b_1, b_2, \ldots, b_d]^T$, where $b_i = mc - n(i-1)$, for $i = 1, \ldots, \lceil p \rceil$ and $b_i = 0$, for $\lceil p \rceil + 1, \ldots, d$. Moreover, let $\mathbf{t} = [t_1, \ldots, t_d]^T$, where t_i is the cardinality of the set U_i in line 6 of the algorithm Greedy or SuperGreedy, for $i = 1, \ldots, d$. We define $r_i = \sum_{j=i}^{d} (jc - i + 1)t_j - b_i$, for $i = 1, 2, \ldots, d$, which is equivalent to

$$\mathbf{A} \cdot \mathbf{t} = \mathbf{b} + \mathbf{r},$$

where $\mathbf{r} = [r_1, \ldots, r_d]^T$. Hence

$$\mathbf{t} = \mathbf{A}^{-1} \cdot (\mathbf{b} + \mathbf{r})$$

and

$$|T| = \mathbf{1} \cdot \mathbf{t} = \mathbf{1} \cdot \mathbf{A}^{-1} \cdot (\mathbf{b} + \mathbf{r}) = \mathbf{1} \cdot \mathbf{A}^{-1} \cdot \mathbf{b} + \mathbf{1} \cdot \mathbf{A}^{-1} \cdot \mathbf{r} = \mathbf{1} \cdot \mathbf{A}^{-1} \cdot \mathbf{b} + \mathbf{e} \cdot \mathbf{r},$$

where $\mathbf{e} = \mathbf{1} \cdot \mathbf{A}^{-1}$.

By Lemma 2.1, $r_1 = 0$ and $r_i \geq 0$, for $i = 2, 3, \ldots, d$. The inequalities (26) in Lemma 3.2 imply now that $\mathbf{e} \cdot \mathbf{r} = \sum_{i=1}^{d} e_i r_i \leq 0$ so

$$|T| \leq \mathbf{1} \cdot \mathbf{A}^{-1} \cdot \mathbf{b}. \quad \square$$

Applying the equality (25) for the matrix $\mathbf{A}$ defined in Theorem 3.3 we get $\mathbf{1} \cdot \mathbf{A}^{-1} = \mathbf{e} = [e_1, e_2 \ldots, e_d]$, where

$$e_j = \frac{(c-1)(j-2)}{(c-1)j+1} e_{j-1}, \tag{28}$$

for $j = 3, \ldots, d$. By Lemma 3.1, $e_1 = \frac{1}{c}$ and $e_2 = -\frac{1}{2c-1}$.

Hence

$$e_j = -\frac{1}{2c-1} \prod_{i=3}^{j} \frac{(c-1)(i-2)}{(c-1)i+1} = -\frac{(c-1)^{j-2}(j-2)!}{\prod_{i=2}^{j}((c-1)i+1)},$$

for $j = 2, \ldots, d$.

By Theorem 3.3

$$|T| \leq \mathbf{1} \cdot \mathbf{A}^{-1} \cdot \mathbf{b} = \mathbf{e} \cdot \mathbf{b} = \sum_{j=1}^{d} e_j b_j.$$

One can readily observe that, as $e_j \leq 0$, when $j \geq 2$, for every positive integer k,

$$\sum_{j=2}^{k} e_j(mc - (j-1)n) \geq \sum_{j=2}^{\lceil \frac{mc}{n} \rceil} e_j(mc - (j-1)n).$$

Thus, as $d \geq \left\lceil \frac{mc}{n} \right\rceil$, for every positive k,

$$|T| \leq \sum_{j=1}^{d} e_j b_j = m + \sum_{j=2}^{\lceil \frac{mc}{n} \rceil} e_j b_j = m + \sum_{j=2}^{\lceil \frac{mc}{n} \rceil} e_j (mc - (j-1)n)$$

$$\leq m + \sum_{j=2}^{k} e_j (mc - (j-1)n) = m - \sum_{j=2}^{k} \frac{(c-1)^{j-2}(j-2)!}{\prod_{i=2}^{j}((c-1)i+1)} (mc - (j-1)n).$$

It can be verified in a standard way that this upper bound for the cardinality of a transversal T coincides with the bound (2) found by Chvátal and McDiarmid [3] by a different approach.

To check it observe that

$$\frac{(c-1)^{j-2}(j-2)!}{\prod_{i=2}^{j}((c-1)i+1)} = H_j - H_{j-1}, \quad \text{where } H_j = -\frac{(c-1)^{j-1}(j-1)!}{\prod_{i=1}^{j}((c-1)i+1)}$$

and

$$\frac{(c-1)^{j-2}(j-1)!}{\prod_{i=2}^{j}((c-1)i+1)} = G_j - G_{j-1}, \quad \text{where } G_j = -\frac{(c-1)^{j-1}j!}{\prod_{i=2}^{j}((c-1)i+1)}.$$

4. Strongly independent sets and Steiner systems

We would like to strengthen inequalities (11) in the case of the SuperGreedy algorithm not only for $i = 2$ (see Lemma 2.4) but for all $i \geq 3$ as well. It will turn out later that when we try to do it, we encounter a problem with Steiner systems.

Let $c \geq 2$. By a *Steiner system* $S(2, c, n)$ we mean a c-uniform hypergraph $H = (X, \mathcal{E})$ such that every pair of vertices in X belongs to exactly one edge in $\mathcal{E}$. In particular, a hypergraph $H = (X, \{X\})$, where $|X| = c$ is a Steiner system $S(2, c, c)$. It is well known that the degrees of vertices in a Steiner system $S(2, c, n)$ are all the same and equal to $k = \frac{n-1}{c-1}$, so Steiner systems $S(2, c, k(c-1)+1)$ are k-regular.

We shall show first that if a connected uniform hypergraph is not a Steiner system then it has a "relatively large" strongly independent set. First we need a simple lemma.

Lemma 4.1. *If no component of a graph G is a complete graph on $\Delta(G)+1$ vertices and $\Delta(G) \geq 3$ then G contains an independent set of cardinality at least $\frac{|G|}{\Delta(G)}$.*

Proof. Brooks' theorem implies that $\chi(G) \leq \Delta(G)$, so one of monochromatic sets in a good coloring of vertices of G into $\Delta(G)$ colors has at least $\frac{|G|}{\Delta(G)}$ elements. □

Lemma 4.2. *Let $c \geq 3$. If a connected c-uniform hypergraph $H = (X, \mathcal{A})$ is not a Steiner system $S(2, c, k(c-1)+1)$ then there is set $U_k \subseteq V_k = \{x \in X : \deg_{\mathcal{A}} x = k\}$, where $k = \Delta(H)$, which is strongly independent in H, such that $|U_k| \geq \frac{|V_k|}{k(c-1)}$.*

Proof. We observe that $k > 1$ because the only c-uniform connected hypergraph with the maximum degree equal to 1 is the Steiner system $S(2, c, c)$.

Let G_k be the graph with the vertex set V_k in which a pair uv, $u \neq v$, is an edge if there is an edge $E \in \mathcal{A}$ such that $u, v \in E$. We observe that $U_k \subseteq V_k$ is a strongly independent set in H if and only if U_k is an independent set in G_k. Obviously, $\Delta(G_k) \leq k(c-1)$. If $\Delta(G_k) < k(c-1)$ then the chromatic number $\chi(G_k)$ of G_k satisfies the inequalities $\chi(G_k) \leq \Delta(G_k) + 1 \leq k(c-1)$ so one of monochromatic sets in a good coloring of vertices of G_k into $k(c-1)$ colors has at least $\frac{|V_k|}{k(c-1)}$ elements. This set is an independent set in G_k and, consequently, strongly independent in H.

We assume now that $\Delta(G_k) = k(c-1) \geq 3$. If no component of G_k is a complete graph on $k(c-1)+1$ vertices then, by Lemma 4.1, $\chi(G_k) \leq k(c-1)$ and we are done as in the previous case.

Suppose now that some component F of G_k is a complete graph on $k(c-1)+1$ vertices. We denote by V the vertex set of F. Let $x \in V$ and assume that some member of $\mathcal{A}_x$ has a vertex which is not a member of V. Then x is a neighbor of at most $k(c-1)-1$ vertices in G_k, a contradiction as $\deg_{G_k} x = k(c-1)$. Thus all edges of H which have a vertex in V are subsets of V. Since the hypergraph H is connected, $X = V$ and the graph G_k is isomorphic to the complete graph on $k(c-1)+1$ vertices. Let $u, v \in X$, $u \neq v$. As uv is an edge in G_k, there is an edge E in H such that $u, v \in E$. If the pair uv belongs to more than one edge in H then $\deg_{G_k} u < k(c-1)$, a contradiction. We have shown that every pair of elements in X belongs to exactly one member of $\mathcal{A}$. Since $|X| = k(c-1)+1$ and all edges in H are c-element, the hypergraph H is a Steiner system $S(2, c, k(c-1)+1)$. This contradiction completes the proof. □

We shall characterize now all maximal strongly independent sets in Steiner systems and hypergraphs obtained from Steiner systems by removing a vertex or all vertices of an edge (together with the incident edges).

Let $H = (X, \mathcal{E})$ be a Steiner system $S(2, c, \ell(c-1)+1)$. For any $x \in X$ and $E \in \mathcal{E}$, we denote $H(x) = (X - \{x\}, \mathcal{E} - \mathcal{E}_x)$, $H(E) = (X - E, \mathcal{E} - \cup_{x\in E} \mathcal{E}_x)$ and $\mathcal{I}_x(H) = \{E - \{x\} : x \in E \in \mathcal{E}_x\}$.

Lemma 4.3. *Let $H = (X, \mathcal{E})$ be a Steiner system $S(2, c, \ell(c-1)+1)$ and let $x \in X$ and $E \in \mathcal{E}$.*

1. *The hypergraph H is ℓ-regular and $\{\{x\} : x \in X\}$ is the family of maximal strongly independent sets in H.*
2. *The hypergraph $H(x)$ is $(\ell-1)$-regular and $\mathcal{I}_x(H)$ is the family of maximal strongly independent sets in $H(x)$.*
3. *The hypergraph $H(E)$ is $(\ell-c)$-regular.*

Proof. 1. By the definition of a Steiner system, every vertex in H belongs to exactly $\frac{\ell(c-1)+1-1}{c-1} = \ell$ edges in H. Moreover, each pair of vertices in H is contained in some edge in H. Therefore maximal strongly independent sets in H are only the one-element subsets of X.

2. For every vertex $y \in X - \{x\}$, there is exactly one edge $E \in \mathcal{E}$ which contains both x and y. Therefore the degree of y in $H(x)$ is $\ell - 1$.

Let $A \in \mathcal{I}_x(H)$ and let $u, v \in A$. By the definition of the Steiner system H, there is a unique edge, say $E \in \mathcal{E}$, such that $u, v \in E$. On the other hand $u, v \in A \cup \{x\} \in \mathcal{E}$ so $E = A \cup \{x\}$. Thus there is no edge in $H(x)$ that contains both u and v because $E = A \cup \{x\} \in \mathcal{E}_x$. We have shown that each member of $\mathcal{I}_x(H)$ is a strongly independent set.

Suppose now that there is a strongly independent set A in $H(x)$ which is not contained in any member of $\mathcal{I}_x(H)$. By the definition of a Steiner system, the members of $\mathcal{I}_x(H)$ form a partition of $X - \{x\}$ (so in particular they are pairwise disjoint). Hence there are $u, v \in A$, such that u and v belong to two different members of $\mathcal{I}_x(H)$. By the definition of a Steiner system once again, there is an edge, say $E \in \mathcal{E}$, such that $u, v \in E$. If $E \in \mathcal{E}_x$ then $u, v \in E - \{x\} \in \mathcal{I}_x(H)$, a contradiction. Thus E is an edge in $H(x)$ so there is no independent set in $H(x)$ containing both u and v. This contradiction implies that every strongly independent set in $H(x)$ is contained in some member of $\mathcal{I}_x(H)$. Since members of $\mathcal{I}_x(H)$ are strongly independent in $H(x)$ and disjoint, $\mathcal{I}_x(H)$ is the family of all maximal strongly independent sets in $H(x)$.

3. Let w be any vertex in $H(E)$. For every vertex $x \in E$ there is a unique edge $E_x \in \mathcal{E}$ such that $x, w \in E$. Moreover, for $x_1, x_2 \in E$, $x_1 \neq x_2$, the edges E_{x_1} and E_{x_2} are different because otherwise, there is an edge in $\mathcal{E}$ containing x_1, x_2 and w. This is, however, impossible because the only edge in $\mathcal{E}$ that contains both x_1 and x_2 is E and $w \notin E$. We have shown that, for every vertex w in $H(E)$, there are $|E| = c$ pairwise different edges in $\cup_{x\in E} \mathcal{E}_x$ that contain w. Thus the degree of every vertex in $H(E)$ is $\ell - c$. □

5. Stronger inequalities for the SuperGreedy algorithm

Lemma 4.3 has some obvious consequences for the algorithm SuperGreedy.

Let us denote by $\mathcal{A}^i$ (resp. T^i) the family $\mathcal{A}$ (resp. the set T) at start of the pass of the "for loop" of the algorithm SuperGreedy $(X, \mathcal{E})$ for $k = i$. Clearly, $\mathcal{A}^i = \mathcal{E} - \cup_{x\in T^i} \mathcal{A}_x$. Finally, let $X^i = X - T^i$ and $H_i = (X^i, \mathcal{A}^i)$.

We recall that we denoted by V_i the set of vertices of degree i in the hypergraph H_i and by U_i a maximum-sized subset of V_i which is strongly independent. We defined $t_i = |U_i|$.

Lemma 5.1. *Suppose that one of the components $F = (Y, \mathcal{B})$ of the hypergraph H_i is a Steiner system $S(2, c, \ell(c-1)+1)$. Let $t'_i = |U_i \cap Y|$, for $i = \ell, \ell-1, \ldots, 1$. Then*

1. *$U_\ell \cap Y = \{x\}$, for some $x \in Y$, so $t'_\ell = 1$,*
2. *$U_{\ell-1} \cap Y = E - \{x\}$, for some $E \in \mathcal{B}$, so $t'_{\ell-1} = c-1$,*
3. *$V_i \cap Y = \emptyset$, for $i = \ell-2, \ell-3, \ldots, \ell-c+1$, so $t'_{\ell-2} = t'_{\ell-3} = \cdots = t'_{\ell-c+1} = 0$.*

Proof. By Lemma 4.3.1, all vertices of F belong to V_ℓ so $U_\ell \cap Y = \{x\}$, for some $x \in Y$. Consequently, $t'_\ell = 1$.

Similarly, by Lemma 4.3.2 all elements of $Y - \{x\}$ belong to $V_{\ell-1}$ and the algorithm SuperGreedy $(X, \mathcal{E})$ selects a set $U_{\ell-1}$ in line 6 of the pass of the "for loop" for $k = \ell - 1$ such that $U_{\ell-1} \cap Y$ is one of the members of $\mathcal{I}_x(F)$, for some $x \in Y$. Clearly, all members of $\mathcal{I}_x(F)$ are $(c-1)$-element sets so $t'_{\ell-1} = c - 1$.

Finally, by Lemma 4.3.3, in the passes of the "for loop" for $i = \ell-2, \ell-3, \ldots, \ell-c+1$ no vertex of F belongs to V_i so $t'_{\ell-2} = t'_{\ell-3} = \cdots = t'_{\ell-c+1} = 0$. □

Let, as before, $p = mc/n$ be the average degree of a vertex in a c-uniform hypergraph $(X, \mathcal{E})$ with n vertices and m edges. Let, for $i = 1, \ldots, \lceil p\rceil$, $X^i_0 \subseteq X^i$ be the set of vertices that belong to components of H_i which are Steiner systems $S(2, c, i(c-1)+1)$. For $i = 1, \ldots, \lceil p\rceil - 1$, we denote by $X^i_1 \subseteq X^i$ the set of vertices of H_i that belong to components of H_{i+1} which are Steiner systems $S(2, c, (i+1)(c-1)+1)$ in H_{i+1}. Finally, for $i = 1, \ldots, \lceil p\rceil - 2$, we denote by $X^i_2 \subseteq X^i$ the set of vertices of H_i that belong to components of H_{i+2} which are Steiner systems $S(2, c, (i+2)(c-1)+1)$ in H_{i+2}. Moreover, we define $X^i_0 = X^{i-1}_1 = X^{i-2}_2 = \emptyset$, for $i > \lceil p\rceil$.

Lemma 5.2. *If $c \geq 3$ then the sets X^i_0, X^i_1 and X^i_2 are pairwise disjoint.*

Proof. Suppose first X^i_0 and X^i_1 have a common element. Then there is a Steiner system $S(2, c, (i+1)(c-1)+1)$, say S_{i+1}, which is a component of H_{i+1} and a Steiner system $S(2, c, i(c-1)+1)$, say S_i, which is a component of H_i such that S_i and S_{i+1} have a common vertex. Hence, obviously, S_i is a subhypergraph of S_{i+1}. We get a contradiction by the definition of a Steiner system. Thus, $X^i_0 \cap X^i_1 = \emptyset$ and, similarly, $X^i_1 \cap X^i_2 = \emptyset$.

As in the preceding paragraph, we observe that if X_0^i and X_2^i have a common element, then there exist a Steiner system $S(2, c, (i+2)(c-1)+1)$, say S_{i+2}, which is a component of H_{i+2} and a Steiner system $S(2, c, i(c-1)+1)$, say S_i, which is a component of H_i such that S_i is a subhypergraph of S_{i+2}. By Lemma 4.3.3, after the deletion of strongly independent vertex sets U_{i+2} and U_{i+1}, each vertex remaining from S_{i+2} is of degree $i-c+2 < i$. This contradiction proves $X_0^i \cap X_2^i = \emptyset$. □

Recall that by d we denote the maximum degree in the hypergraph $(X, \mathcal{E})$. Let us define the sequence s_i, $i = 1, 2, \ldots, d$, in the following way

$$s_i = \begin{cases} \text{the number of components of the hypergraph} \\ H_i = (X^i, \mathcal{A}^i) \text{ which are Steiner} \\ \text{systems } S(2, c, i(c-1)+1) & \text{for } i = 1, \ldots, \lceil p \rceil \\ 0 & \text{for } i > \lceil p \rceil. \end{cases} \tag{29}$$

We are ready now to strengthen the inequalities (11) in the case of the algorithm SuperGreedy.

Lemma 5.3. *Let $c \geq 3$ and let t_i, $i = d, d-1, \ldots, 1$, be the cardinality of the set U_i occurring in the algorithm* SuperGreedy *applied for a c-uniform hypergraph with n vertices, m edges and maximum degree equal to d. For $i = 1, 2, \ldots, \lceil p \rceil$,*

$$\beta(i,c)t_i + \sum_{j=i+1}^{d} (jc - i + 1)t_j \geq mc - n(i-1) - ((c-1)i + 1 - \beta(i,c))s_i + (c-1)(\beta(i,c) - i - 1)s_{i+1} + (c-1)(c-3)(i+1)s_{i+2}, \tag{30}$$

where $p = \frac{mc}{n}$ and

$$\beta(i,c) = \begin{cases} i(c-1) & \text{for } 1 \leq i \leq \lceil p \rceil, i \neq 2 \\ \lfloor 3c/2 \rfloor & \text{for } i = 2. \end{cases} \tag{31}$$

Proof. By Lemma 5.1, for $i = 1, \ldots, \lceil p \rceil$,

$$\begin{aligned} |X_0^i| &= s_i((c-1)i + 1), \\ |X_1^i| &= s_{i+1}(c-1)(i+1), \\ |X_2^i| &= s_{i+2}((c-1)(i+2) + 1 - c) = s_{i+2}(c-1)(i+1) \end{aligned} \tag{32}$$

and

$$\begin{aligned} |U_i \cap X_0^i| &= s_i, \\ |U_i \cap X_1^i| &= (c-1)s_{i+1}, \\ |U_i \cap X_2^i| &= 0. \end{aligned} \tag{33}$$

Let $X_3^i = X^i - (X_0^i \cup X_1^i \cup X_2^i)$. By Lemmas 4.2 and 2.3, we find in line 6 in the pass of the "for loop" of the algorithm SuperGreedy $(X, \mathcal{E})$ for $k = i$, a maximum-sized subset $U_i \subseteq V_i$ which is strongly independent in $\mathcal{A}^i$ such that

$$|U_i \cap X_3^i| \geq \frac{|V_i \cap X_3^i|}{\beta(i,c)}, \quad \text{for } i = 1, \ldots, \lceil p \rceil, \tag{34}$$

By the inequality (34), Lemma 5.2 and the equalities (33), for $i = 1, \ldots, \lceil p \rceil$, we get

$$\begin{aligned} |V_i \cap X_3^i| \leq \beta(i,c)|U_i \cap X_3^i| &= \beta(i,c)(|U_i| - |U_i \cap X_0^i| - |U_i \cap X_1^i| - |U_i \cap X_2^i|) \\ &= \beta(i,c)(t_i - s_i - (c-1)s_{i+1}). \end{aligned} \tag{35}$$

Clearly,

$$|X_3^i| = n - \sum_{j=i+1}^{d} t_j - |X_0^i| - |X_1^i| - |X_2^i|$$

and

$$\begin{aligned} \sum_{x \in X_3^i} \deg_{\mathcal{A}^i} x &\leq i|V_i \cap X_3^i| + (i-1)\left(n - \sum_{j=i+1}^{d} t_j - |X_0^i| - |X_1^i| - |X_2^i| - |V_i \cap X_3^i|\right) \\ &= |V_i \cap X_3^i| - (i-1)(|X_0^i| + |X_1^i| + |X_2^i|) + (i-1)\left(n - \sum_{j=i+1}^{d} t_j\right). \end{aligned} \tag{36}$$

By Lemma 4.3 and (36), the degrees in H_i of vertices in $X_0^i \cup X_1^i$ are equal to i and the degrees in H_i of vertices in X_2^i are equal to $i+2-c$. Therefore

$$\begin{aligned}\sum_{x\in X^i}\deg_{\mathcal{A}^i} x &= \sum_{x\in X_0^i\cup X_1^i}\deg_{\mathcal{A}^i} x + \sum_{x\in X_2^i}\deg_{\mathcal{A}^i} x + \sum_{x\in X_3^i}\deg_{\mathcal{A}^i} x\\ &= i(|X_0^i|+|X_1^i|) + (i+2-c)|X_2^i| + \sum_{x\in X_3^i}\deg_{\mathcal{A}^i} x\\ &\le |X_0^i|+|X_1^i|+(3-c)|X_2^i|+|V_i\cap X_3^i|+(i-1)\left(n-\sum_{j=i+1}^{d}t_j\right). \end{aligned} \tag{37}$$

By the inequalities (37), (35) and the equalities (32), for $i=1,\ldots,\lceil p\rceil$, we get

$$\begin{aligned}\sum_{x\in X^i}\deg_{\mathcal{A}^i} x &\le s_i((c-1)i+1)+s_{i+1}(c-1)(i+1)+(3-c)s_{i+2}(c-1)(i+1)\\ &\quad+\beta(i,c)(t_i-s_i-(c-1)s_{i+1})+(i-1)\left(n-\sum_{j=i+1}^{d}t_j\right)\\ &= ((c-1)i+1-\beta(i,c))s_i+(c-1)(i+1-\beta(i,c))s_{i+1}-(c-1)(c-3)(i+1)s_{i+2}\\ &\quad+\beta(i,c)t_i+(i-1)\left(n-\sum_{j=i+1}^{d}t_j\right). \end{aligned} \tag{38}$$

On the other hand

$$\sum_{x\in X^i}\deg_{\mathcal{A}^i} x = mc-\sum_{j=i+1}^{d} jct_j. \tag{39}$$

By (38) and (39), for $i=1,\ldots,\lceil p\rceil$,

$$\begin{aligned} mc-\sum_{j=i+1}^{d} jct_j &\le \beta(i,c)t_i+(i-1)\left(n-\sum_{j=i+1}^{d}t_j\right)\\ &\quad+((c-1)i+1-\beta(i,c))s_i+(c-1)(i+1-\beta(i,c))s_{i+1}-(c-1)(c-3)(i+1)s_{i+2}, \end{aligned}$$

which is equivalent to (30). □

6. Main results

For the algorithm SuperGreedy, we would like to strengthen the inequalities (11) to

$$(ic-i)t_i+\sum_{j=i+1}^{d}(jc-i+1)t_j \ge mc-n(i-1) \tag{40}$$

for $i=2,3,\ldots,d$. Unfortunately it cannot be done. For example the inequality (40), for $i=d$, is false when the input hypergraph is a Steiner system $S(2,c,d(c-1)+1)$.

Nevertheless, using Lemma 5.3, we will be able to prove that there exist real numbers $t'_1,\ldots,t'_d$ which satisfy the inequalities (40) and such that $t'_1+\cdots+t'_d=t_1+\cdots+t_d=|T|$, where T is a transversal constructed by the algorithm SuperGreedy. This leads to a stronger analogue of Theorem 3.3 for this algorithm with the matrix $\mathbf{A}$ replaced by $\mathbf{A}'$ (defined below).

Let, for $c\ge 4$, $\mathbf{A}'$ be a $d\times d$ matrix such that

$$a'_{i,j}=\begin{cases} cj-i+1 & \text{if } j>i\\ 0 & \text{if } j<i\\ cj-i & \text{if } i=j \text{ and } i\ge 3\\ c & \text{if } i=j=1\\ \lfloor 3c/2\rfloor & \text{if } i=j=2.\end{cases} \tag{41}$$

For $c=3$, we define $\mathbf{A}'$ as in the case of $c\ge 4$ except for the entry $a'_{4,4}$ which is equal to 9 (i.e. larger by 1 than $a_{4,4}$ in $\mathbf{A}$).

In other words, for $c \neq 3$,

$$\mathbf{A}' = \begin{bmatrix} c & 2c & 3c & 4c & 5c & \cdots & dc \\ 0 & \lfloor 3c/2 \rfloor & 3c-1 & 4c-1 & 5c-1 & \cdots & dc-1 \\ 0 & 0 & 3c-3 & 4c-2 & 5c-2 & \cdots & dc-2 \\ 0 & 0 & 0 & 4c-4 & 5c-3 & \cdots & dc-3 \\ 0 & 0 & 0 & 0 & 5c-5 & \cdots & dc-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & (d-1)c-(d-1) & dc-(d-2) \\ 0 & 0 & 0 & 0 & \cdots & 0 & dc-d \end{bmatrix}$$

and, for $c = 3$,

$$\mathbf{A}' = \begin{bmatrix} c & 2c & 3c & 4c & 5c & \cdots & dc \\ 0 & \lfloor 3c/2 \rfloor & 3c-1 & 4c-1 & 5c-1 & \cdots & dc-1 \\ 0 & 0 & 3c-3 & 4c-2 & 5c-2 & \cdots & dc-2 \\ 0 & 0 & 0 & 4c-3 & 5c-3 & \cdots & dc-3 \\ 0 & 0 & 0 & 0 & 5c-5 & \cdots & dc-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & (d-1)c-(d-1) & dc-(d-2) \\ 0 & 0 & 0 & 0 & \cdots & 0 & dc-d \end{bmatrix}.$$

Theorem 6.1. *Let $c \geq 3$. For a c-uniform hypergraph with n vertices, m edges and the maximum degree equal to d, the cardinality of the transversal T constructed by the algorithm* SuperGreedy *satisfies the inequality*

$$|T| \leq \mathbf{1} \cdot (\mathbf{A}')^{-1} \cdot \mathbf{b},$$

where $\mathbf{b} = [mc, mc-n, mc-2n, \ldots, mc-(\lceil p \rceil - 2)n, mc-(\lceil p \rceil - 1)n, 0, 0, \ldots, 0]^T$, $\mathbf{1} = [1, 1, \ldots, 1]$ *and* $p = \frac{mc}{n}$.

The next technical lemma explains how Theorem 6.1 will be proved.

Lemma 6.2. *Let $c \geq 3$ and let $t_i, i = 1, \ldots, d$, be the cardinalities of the sets U_i constructed in line 6 of the algorithm* SuperGreedy. *There exist real numbers $t'_1, \ldots, t'_d$ such that*

(i) $\sum_{i=1}^{d} t'_i = \sum_{i=1}^{d} t_i$,
(ii) $\sum_{j=1}^{d} cjt'_j = mc$,
(iii) $\lfloor 3c/2 \rfloor t'_2 + \sum_{j=3}^{d} (jc-1)t'_j \geq mc - n$,
(iv) $i(c-1)t'_i + \sum_{j=i+1}^{d} (jc-i+1)t'_j \geq mc-(i-1)n$, *for* $i = 3, 4 \ldots, \lceil p \rceil$ *except* $i = 4$ *and* $c = 3$,
(v) $\sum_{j=i}^{d} (jc-i+1)t'_j \geq mc-(i-1)n$, *for* $i = 4 \leq \lceil p \rceil$ *and* $c = 3$,
(vi) $i(c-1)t'_i + \sum_{j=i+1}^{d} (jc-i+1)t'_j \geq 0$, *for* $i = \lceil p \rceil + 1, \ldots, d$.

Proof. We define $\alpha_j, j = 1, \ldots, d$, $\beta_j, j = 2, \ldots, d+1$, $\gamma_j, j = 3, \ldots, d+2$, and $\delta_j, j = 4, \ldots, d+3$ as follows.

For $c \geq 4$ and $j = 4, 5, \ldots, d$ or $c = 3$ and $j = 5, 6, \ldots, d$,

$$\alpha_j = \frac{1}{2}, \qquad \beta_j = -\frac{3}{2}, \qquad \gamma_j = \frac{3}{2}, \qquad \delta_j = -\frac{1}{2}. \tag{42}$$

For $c = 3$,

$$\alpha_4 = \beta_4 = \gamma_4 = \delta_4 = 0. \tag{43}$$

Finally, for all $c \geq 3$,

$$\alpha_3 = 1, \qquad \beta_3 = -2, \qquad \gamma_3 = 1, \tag{44}$$

$$\alpha_2 = \beta_2 = 0, \tag{45}$$

$$\alpha_1 = 0. \tag{46}$$

Clearly, for $j = 1, 2, \ldots, d$,

$$\alpha_j + \beta_j + \gamma_j + \delta_j = 0, \tag{47}$$

and, for $j = 4, 5, \ldots, d$,

$$j\alpha_j + (j-1)\beta_j + (j-2)\gamma_j + (j-3)\delta_j = 0. \tag{48}$$

For $j = 1, 2, \ldots, d$, we define

$$t'_j = t_j + \alpha_j s_j + \beta_{j+1}s_{j+1} + \gamma_{j+2}s_{j+2} + \delta_{j+3}s_{j+3}. \tag{49}$$

(The numbers s_i are defined by the equality (29).)

By (47) and the fact that $s_j = 0$, for $j > d \geq \lceil p \rceil$ and $i = 1, \ldots, d$, we get

$$\begin{aligned}
\sum_{j=i}^{d} t'_j - \sum_{j=i}^{d} t_j &= \sum_{j=i}^{d}(t'_j - t_j) = \sum_{j=i}^{d}(\alpha_j s_j + \beta_{j+1}s_{j+1} + \gamma_{j+2}s_{j+2} + \delta_{j+3}s_{j+3}) \\
&= \sum_{j=i}^{d} \alpha_j s_j + \sum_{j=i+1}^{d+1} \beta_j s_j + \sum_{j=i+2}^{d+2} \gamma_j s_j + \sum_{j=i+3}^{d+3} \delta_j s_j \\
&= \sum_{j=i}^{i+2} \alpha_j s_j + \sum_{j=i+1}^{i+2} \beta_j s_j + \sum_{j=i+2}^{i+2} \gamma_j s_j + \sum_{j=i+3}^{d} (\alpha_j + \beta_j + \gamma_j + \delta_j)s_j \\
&= \alpha_i s_i + (\alpha_{i+1} + \beta_{i+1})s_{i+1} + (\alpha_{i+2} + \beta_{i+2} + \gamma_{i+2})s_{i+2}.
\end{aligned} \tag{50}$$

Applying (50) for $i = 1$, by (44)–(46), we get the equality (i).

Similarly, by (48) and the fact that $s_i = 0$, for $j > d$ and $i = 1, \ldots, d$,

$$\begin{aligned}
\sum_{j=i}^{d} jt'_j - \sum_{j=i}^{d} jt_j &= \sum_{j=i}^{d} j(t'_j - t_j) = \sum_{j=i}^{d}(j\alpha_j s_j + j\beta_{j+1}s_{j+1} + j\gamma_{j+2}s_{j+2} + j\delta_{j+3}s_{j+3}) \\
&= \sum_{j=i}^{d} j\alpha_j s_j + \sum_{j=i+1}^{d+1} (j-1)\beta_j s_j + \sum_{j=i+2}^{d+2} (j-2)\gamma_j s_j + \sum_{j=i+3}^{d+3} (j-3)\delta_j s_j \\
&= \sum_{j=i}^{i+2} j\alpha_j s_j + \sum_{j=i+1}^{i+2} (j-1)\beta_j s_j + \sum_{j=i+2}^{i+2} (j-2)\gamma_j s_j \\
&\quad + \sum_{j=i+3}^{d} (j\alpha_j + (j-1)\beta_j + (j-2)\gamma_j + (j-3)\delta_j)s_j \\
&= i\alpha_i s_i + ((i+1)\alpha_{i+1} + i\beta_{i+1})s_{i+1} + ((i+2)\alpha_{i+2} + (i+1)\beta_{i+2} + i\gamma_{i+2})s_{i+2}.
\end{aligned} \tag{51}$$

Applying (51) for $i = 1$, by (44)–(46), we get

$$\sum_{j=1}^{d} jt'_j - \sum_{j=1}^{d} jt_j = \alpha_1 s_1 + (2\alpha_2 + \beta_2)s_2 + (3\alpha_3 + 2\beta_3 + \gamma_3)s_3 = 0. \tag{52}$$

The equality (ii) follows now from Lemma 2.1.

To prove (iii), by (50), (51) and (42)–(45), we get

$$\begin{aligned}
&\lfloor 3c/2 \rfloor t'_2 + \sum_{j=3}^{d} (jc-1)t'_j - \left(\lfloor 3c/2 \rfloor t_2 + \sum_{j=3}^{d} (jc-1)t_j \right) \\
&= c\left(\sum_{j=3}^{d} jt'_j - \sum_{j=3}^{d} jt_j \right) - \left(\sum_{j=3}^{d} t'_j - \sum_{j=3}^{d} t_j \right) + \lfloor 3c/2 \rfloor (t'_2 - t_2) \\
&= c(3\alpha_3 s_3 + (4\alpha_4 + 3\beta_4)s_4 + (5\alpha_5 + 4\beta_5 + 3\gamma_5)s_5) \\
&\quad - (\alpha_3 s_3 + (\alpha_4 + \beta_4)s_4 + (\alpha_5 + \beta_5 + \gamma_5)s_5) + \lfloor 3c/2 \rfloor (\alpha_2 s_2 + \beta_3 s_3 + \gamma_4 s_4 + \delta_5 s_5) \\
&= \lfloor 3c/2 \rfloor \alpha_2 s_2 + [(3c-1)\alpha_3 + \lfloor 3c/2 \rfloor \beta_3]s_3 + [(4c-1)\alpha_4 + (3c-1)\beta_4 + \lfloor 3c/2 \rfloor \gamma_4]s_4 \\
&\quad + [(5c-1)\alpha_5 + (4c-1)\beta_5 + (3c-1)\gamma_5 + \lfloor 3c/2 \rfloor \delta_5]s_5 \\
&= (c - 1 - 2\lfloor c/2 \rfloor)s_3 + [(4c-1)\alpha_4 + (3c-1)\beta_4 + \lfloor 3c/2 \rfloor \gamma_4]s_4 + \left(c - \frac{1}{2} - \frac{1}{2}\lfloor 3c/2 \rfloor \right) s_5 \\
&\geq \begin{cases} -s_3 - \frac{1}{4}cs_4 & \text{for } c \geq 4 \\ 0 & \text{for } c = 3. \end{cases}
\end{aligned} \tag{53}$$

Hence, by Lemma 5.3, for $c \geq 4$,

$$\lfloor 3c/2 \rfloor t_2' + \sum_{j=3}^{d}(jc-1)t_j' \geq mc - n - (c - 1 - \lfloor c/2 \rfloor)s_2 + [(c-1)(\lfloor 3c/2 \rfloor - 3) - 1]s_3 + \left[3(c-1)(c-3) - \frac{1}{4}c\right] s_4 \geq mc - n,$$

because $s_2 = 0$, as Steiner systems $S(2, c, 2c-1)$ do not exist.

Similarly, by (53),

$$\lfloor 3c/2 \rfloor t_2' + \sum_{j=3}^{d}(jc-1)t_j' \geq mc - n,$$

for $c = 3$, which completes the proof of (iii).

To prove (iv), by (50) and (51), we get

$$\begin{aligned}
&i(c-1)t_i' + \sum_{j=i+1}^{d}(jc-i+1)t_j' - \left(i(c-1)t_i + \sum_{j=i+1}^{d}(jc-i+1)t_j\right) \\
&= c\left(\sum_{j=i}^{d} jt_j' - \sum_{j=i}^{d} jt_j\right) - i\left(\sum_{j=i}^{d} t_j' - \sum_{j=i}^{d} t_j\right) + \left(\sum_{j=i+1}^{d} t_j' - \sum_{j=i+1}^{d} t_j\right) \\
&= c(i\alpha_i s_i + ((i+1)\alpha_{i+1} + i\beta_{i+1})s_{i+1} + ((i+2)\alpha_{i+2} + (i+1)\beta_{i+2} + i\gamma_{i+2})s_{i+2}) - i(\alpha_i s_i + (\alpha_{i+1} + \beta_{i+1})s_{i+1} \\
&\quad + (\alpha_{i+2} + \beta_{i+2} + \gamma_{i+2})s_{i+2}) + \alpha_{i+1}s_{i+1} + (\alpha_{i+2} + \beta_{i+2})s_{i+2} + (\alpha_{i+3} + \beta_{i+3} + \gamma_{i+3})s_{i+3} \\
&= (c-1)i\alpha_i s_i + [((c-1)i + c + 1)\alpha_{i+1} + (c-1)i\beta_{i+1}]s_{i+1} \\
&\quad + [((c-1)i + 2c + 1)\alpha_{i+2} + ((c-1)i + c + 1)\beta_{i+2} + (c-1)i\gamma_{i+2}]s_{i+2} + (\alpha_{i+3} + \beta_{i+3} + \gamma_{i+3})s_{i+3}.
\end{aligned}$$

Hence, by Lemma 5.3, as $i \geq 3$,

$$\begin{aligned}
i(c-1)t_i' + \sum_{j=i+1}^{d}(jc-i+1)t_j' &\geq mc - n(i-1) + [(c-1)i\alpha_i - 1]s_i + [((c-1)i + c + 1)\alpha_{i+1} + (c-1)i\beta_{i+1} \\
&\quad + (c-1)(ci - 2i - 1)]s_{i+1} + [((c-1)i + 2c + 1)\alpha_{i+2} + ((c-1)i + c + 1)\beta_{i+2} + (c-1)i\gamma_{i+2} + (c-1) \\
&\quad \times (c-3)(i+1)]s_{i+2} + (\alpha_{i+3} + \beta_{i+3} + \gamma_{i+3})s_{i+3}.
\end{aligned}$$

If $c \geq 4$ and $i = 4, 5, \ldots, \lceil p \rceil$ or $c = 3$ and $i = 5, 6, \ldots, \lceil p \rceil$ then, by (42),

$$\begin{aligned}
i(c-1)t_i' + \sum_{j=i+1}^{d}(jc-i+1)t_j' - (mc - n(i-1)) &\geq \left[\frac{1}{2}(c-1)i - 1\right]s_i + \left[(c-3)\left(i(c-1) - \frac{1}{2}\right)\right]s_{i+1} \\
&\quad + \left[\frac{1}{2}((c-1)i - (c+2)) + (c-1)(c-3)(i+1)\right]s_{i+2} \\
&\quad + \frac{1}{2}s_{i+3} \geq 0.
\end{aligned}$$

If $i = 3$ then, similarly, by (43)–(46)

$$\begin{aligned}
&3(c-1)t_3' + \sum_{j=4}^{d}(jc-2)t_j' - (mc - 2n) \\
&\geq \begin{cases} (3c-4)s_3 + \left[(c-1)(3c-10) + \frac{1}{2}(c+1)\right]s_4 + \left[c - \frac{5}{2} + 4(c-1)(c-3)\right]s_5 + \frac{1}{2}s_6 & \text{if } c \geq 4 \\ 5s_3 + 4s_4 + \frac{1}{2}s_5 + \frac{1}{2}s_6 & \text{if } c = 3 \end{cases} \\
&\geq 0.
\end{aligned}$$

This completes the proof of (iv).

Applying (50) and (51) for $i = 4$ and $c = 3$, we get

$$\sum_{j=4}^{d}(3j-3)t'_j - \sum_{j=4}^{d}(3j-3)t_j = -\frac{15}{2}s_5 + 3s_6 \geq 0 \tag{54}$$

because $s_5 = 0$, as Steiner systems $S(2, 3, 11)$ do not exist.

The inequality (v) follows now from (54) and Lemma 2.1.

Since $s_i = 0$, for $i = \lceil p \rceil + 1, \ldots, d$, by (49), $t_i = t'_i$, for $i = \lceil p \rceil + 1, \ldots, d$. Hence,

$$i(c-1)t'_i + \sum_{j=i+1}^{d}(jc-i+1)t'_j = i(c-1)t_i + \sum_{j=i+1}^{d}(jc-i+1)t_j \geq 0,$$

so the inequality (vi) holds. □

Proof of Theorem 6.1. Let $t'_1, \ldots, t'_d$ be a sequence of real numbers whose existence is guaranteed by Lemma 6.2. We define

$$r_i = \begin{cases} \lfloor 3c/2 \rfloor t'_2 + \sum_{j=3}^{d}(jc-1)t'_j - mc + n & \text{if } i = 2 \\ i(c-1)t'_i + \sum_{j=i+1}^{d}(jc-i+1)t'_j - mc + n(i-1) & \text{if } i = 3, 4, \ldots, \lceil p \rceil \text{ and } (i \neq 4 \text{ or } c \neq 3) \\ \sum_{j=i}^{d}(jc-i+1)t'_j - mc + n(i-1) & \text{if } i = 1 \text{ or } (i = 4 \text{ and } c = 3) \\ i(c-1)t'_i + \sum_{j=i+1}^{d}(jc-i+1)t'_j & \text{if } i = \lceil p \rceil + 1, \ldots, d. \end{cases} \tag{55}$$

Let $\mathbf{t}' = [t'_1, \ldots, t'_d]^T$ and $\mathbf{r} = [r_1, \ldots, r_d]^T$. The definition (55) can now be written as

$$\mathbf{r} = \mathbf{A}' \cdot \mathbf{t}' - \mathbf{b}. \tag{56}$$

The conditions (ii)–(vi) in Lemma 6.2 are equivalent to

$$r_i \geq 0, \tag{57}$$

for $i = 2, \ldots, d$ and

$$r_1 = 0. \tag{58}$$

Applying Lemma 6.2(i) we get

$$\mathbf{1} \cdot \mathbf{t}' = \sum_{j=1}^{d} t'_j = \sum_{j=1}^{d} t_j = \mathbf{1} \cdot \mathbf{t},$$

so, by (56),

$$|T| = \mathbf{1} \cdot \mathbf{t} = \mathbf{1} \cdot \mathbf{t}' = \mathbf{1} \cdot (\mathbf{A}')^{-1} \cdot (\mathbf{b} + \mathbf{r}) = \mathbf{1} \cdot (\mathbf{A}')^{-1} \cdot \mathbf{b} + \mathbf{1} \cdot (\mathbf{A}')^{-1} \cdot \mathbf{r} = \mathbf{1} \cdot (\mathbf{A}')^{-1} \cdot \mathbf{b} + \mathbf{e} \cdot \mathbf{r},$$

where $\mathbf{e} = \mathbf{1} \cdot (\mathbf{A}')^{-1}$. Lemma 3.2, (57) and (58) imply that $\mathbf{e} \cdot \mathbf{r} = \sum_{i=1}^{d} e_i r_i \leq 0$ so

$$|T| \leq \mathbf{1} \cdot (\mathbf{A}')^{-1} \cdot \mathbf{b}. \quad \square$$

We are now ready to formulate the main results of this section.

Theorem 6.3. *Let $c \geq 3$ and let $p_j, j = 1, 2, \ldots$, be the sequence defined by the equalities (4)–(9). Then the cardinality of the transversal T constructed by the algorithm* SuperGreedy *applied for a c-uniform hypergraph with n vertices and m edges satisfies the inequality*

$$|T| \leq \left(c \sum_{i=1}^{k} p_i\right) m - \left(\sum_{i=1}^{k}(i-1)p_i\right) n, \tag{59}$$

for every $k = 1, 2, \ldots$.

Proof. The matrix $\mathbf{A}'$ in Theorem 6.1 satisfies the premises of Lemmas 3.1 and 3.2 so $\mathbf{1}\cdot(\mathbf{A}')^{-1}=\mathbf{e}$, where $\mathbf{e}$ is defined by the equalities (23) and (25). Applying these lemmas for the matrix $\mathbf{A}'$ we see that $e_i = p_i$, for $i = 1, \ldots, d$, where d is the maximum degree of the hypergraph. By Theorem 6.1,

$$|T| \le \mathbf{e}\cdot\mathbf{b} = \sum_{i=1}^{\lceil p\rceil} e_i b_i,$$

where $p = \frac{mc}{n}$.

As $b_i = mc - (i-1)n \ge 0$, for $i \le \lceil p\rceil$ and $e_i \le 0$, for $i \ge 2$ (by Lemma 3.2), we get

$$|T| \le \sum_{i=1}^{\lceil p\rceil} e_i b_i \le \sum_{i=1}^{k} e_i b_i = \sum_{i=1}^{k} p_i b_i = \left(c\sum_{i=1}^{k} p_i\right) m - \left(\sum_{i=1}^{k}(i-1)p_i\right) n,$$

for every $k = 1, 2, \ldots, \lceil p\rceil$. Validity of the inequality (59) for $k > \lceil p\rceil$ follows from its validity for $k = \lceil p\rceil$ and the observation that $(mc-(i-1)n)p_i \ge 0$, for $i > \lceil p\rceil$. □

Theorem 1.1 is now an obvious consequence of Theorem 6.3.

Here are some special cases of Theorem 1.1.

Let H be a c-uniform hypergraph with n vertices and m edges. For $k = 1$ Theorem 1.1 gives the trivial inequality $\tau(H) \le m$. For $k = 2$ as well as for $k = 3$ and c odd our Theorem 1.1 gives the inequality (1) proved in [3]. For $k = 4$ and $c = 3$ we get the inequality $\tau(H) \le \frac{1}{6}m + \frac{1}{3}n$ that was shown in [3] too.

In the case of $c = 3$ and $k = 5, 6$ we get the following results.

Corollary 6.4. *If H is a 3-uniform hypergraph with n vertices and m edges then*

$$\tau(H) \le \frac{7}{60}m + \frac{2}{5}n,$$
$$\tau(H) \le \frac{11}{120}m + \frac{53}{120}n. \quad \square$$

These inequalities and, generally, all inequalities (10) for $c = 3$ and $k \ge 5$ are, to our best knowledge, new.

Applying Theorem 1.1 for $k = 3, 4$, when $c \ge 4$ is even and for $k = 4, 5$, when $c \ge 5$ is odd, we get the following corollaries.

Corollary 6.5. *Let $c \ge 4$ be even. If H is a c-uniform hypergraph with n vertices and m edges then*

$$\tau(H) \le \frac{3c-5}{9(c-1)}m + \frac{2(3c-1)}{9c(c-1)}n,$$
$$\tau(H) \le \frac{9c^2-27c+22}{36(c-1)^2}m + \frac{33c^2-47c+2}{36c(c-1)^2}n.$$

Corollary 6.6. *Let $c \ge 5$ be odd. If H is a c-uniform hypergraph with n vertices and m edges then*

$$\tau(H) \le \frac{3c-4}{4(3c-1)}m + \frac{11}{4(3c-1)}n,$$
$$\tau(H) \le \frac{6c^2-15c+10}{10(c-1)(3c-1)}m + \frac{67c-75}{20(c-1)(3c-1)}n. \quad \square$$

For $c = 4, 5, 6$ the inequalities given in Corollaries 6.5 and 6.6 reduce to the inequalities listed below.

Corollary 6.7. *Let H be a c-uniform hypergraph with n vertices and m edges.*

(i) *If $c = 4$ then*

$$\tau(H) \le \frac{7}{27}m + \frac{11}{54}n,$$
$$\tau(H) \le \frac{29}{162}m + \frac{19}{72}n.$$

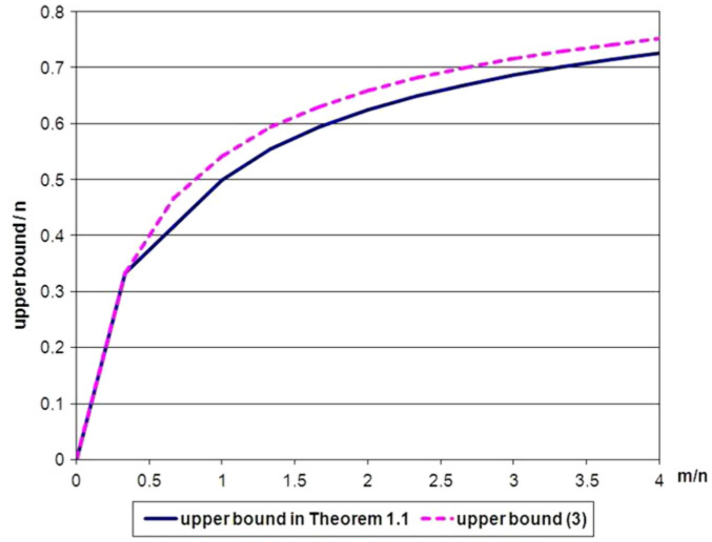

Fig. 1. A comparison of the upper bounds for the transversal number when $c = 3$.

(ii) *If* $c = 5$ *then*

$$\tau(H) \leq \frac{11}{56}m + \frac{11}{56}n,$$
$$\tau(H) \leq \frac{17}{112}m + \frac{13}{56}n. \tag{60}$$

(iii) *If* $c = 6$ *then*

$$\tau(H) \leq \frac{13}{45}m + \frac{17}{135}n,$$
$$\tau(H) \leq \frac{46}{225}m + \frac{227}{1350}n. \quad \square$$

7. Concluding remarks

We observe that if $(\ell-1)\frac{n}{c} \leq m \leq \ell\frac{n}{c}$ then the expression on the right hand side of the inequality (10) in Theorem 1.1 is the least for $k = \ell$. Indeed, $cm - (i-1)n > 0$ if and only if $i \leq \ell$ and, by Lemma 3.2, e_i is positive for $i = 1$ and nonpositive for $i \geq 2$. Hence the terms of the sum $\sum(cm - (i-1)n)e_i$ are nonpositive for $1 < i \leq \ell$ and nonnegative for $i > \ell$.

In Fig. 1 we compare, for $c = 3$, the upper bounds for the transversal number obtained from the inequalities (10) using the observation above with the upper bounds (3) proved in [3]. The horizontal axis represents the ratio $\frac{m}{n}$ and the vertical axis represents the ratio $\frac{\text{an upper bound for } g(3,m,n)}{n}$. (We recall that $g(c, m, n)$ is the smallest integer such that every c-uniform hypergraph H with n vertices and m edges has $\tau(H) \leq g(c, m, n)$.) For $\frac{1}{2} \leq \frac{m}{n} \leq \frac{4}{3}$ our bound obtained by (10) coincides with the bounds proved by Tuza [9] and Chvátal and McDiarmid [3]. For $\frac{m}{n} > \frac{4}{3}$ our bound improves over previously known bounds.

For $c = 4$ the upper bounds given in Corollary 6.5 are our best upper bounds for $\frac{n}{2} \leq m \leq n$. They are, however, worse than the best linear upper bound $g(4, m, n) \leq \frac{4m+5n}{21}$ for this interval, proved in [7]. We have checked that the upper bound for $g(4, m, n)$ proved in Theorem 1.1 is smaller than the bound $\frac{4m+5n}{21}$, for $m \geq 1.2606\ldots n$.

There is a close relationship between the transversal numbers of a c-uniform hypergraphs and the total domination numbers of graphs with minimum degree at least c. By the *total dominating set* we mean a subset S of the sets of vertices of a graph G such that every vertex in G has a neighbor in S. The minimum cardinality of a total dominating set is called the *total domination number* of G and denoted by $\gamma_t(G)$. Total dominating sets have received a considerable attention by researchers in the recent years (e.g. see the survey paper by Henning [4]). It is easy to observe (e.g. see [7] or [4]) that for a graph G with n vertices and the minimum degree at least c

$$\gamma_t(G) \leq g(c, n, n).$$

For $c = 1, 2, 3, 4$ tight upper bounds for $\gamma_t(G)$, where G is a graph with n vertices and the minimum degree at least c, are known (see [4]). For $c = 5$ such a tight bound is not known.

In [7] the authors conjecture that $g(5, n, n) \leq \frac{4}{11}n = 0.3636\ldots n$ and show that, if true, this bound is tight. It follows from our inequality (60) that $g(5, n, n) \leq \frac{43}{112}n = 0.3839\ldots n$. For comparison, the general inequality (2) proved in [3] gives $g(5, n, n) \leq 0.4116\ldots n$.

As a large amount of effort has been done on finding or estimating $g(c, n, n)$ for small values of c (especially in the context of total domination of graphs), below we give the upper bounds on $g(c, n, n)$, for $c = 6, 7, 8$, that follow from Theorem 1.1:

$$g(6, n, n) \leq 0.3523\ldots n,$$
$$g(7, n, n) \leq 0.3180\ldots n,$$
$$g(8, n, n) \leq 0.2969\ldots n.$$

These bounds are obtained by applying Theorem 1.1 for $k = c$.

Finally, let us point out that there is a direct connection between our results and linear upper bounds on domination numbers of c-uniform hypergraphs. A *dominating set* in a hypergraph H is a subset D of the set of vertices of H such that for every vertex $v \in V - D$ there exists an edge E in H for which $v \in E$ and $E \cap D \neq \emptyset$. The *domination number* $\gamma(H)$ is the minimum cardinality of a dominating set in H. Bujtás, Henning, and Tuza [2] proved that for any two nonnegative real numbers a and b and for every $c \geq 3$, if all $(c - 1)$-uniform hypergraphs H satisfy the inequality $\tau(H) \leq am + bn$, then all c-uniform hypergraphs H satisfy the inequality $\gamma(H) \leq (a - b)m + bn$. Due to this result, our Theorem 1.1, also implies some upper bounds for the domination number $\gamma(H)$.

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