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EMBEDDING MODES INTO SEMIMODULES, PART II

Abstract. The first part of this paper specified the semi-affinization semiring of a mode variety as the universal scalar semiring for semimodules whose idempotent subreducts lie in the given variety of modes. The current part of the paper focusses on some selected varieties of modes (affine spaces, barycentric algebras, semilattice modes), and computes the semi-affinization semirings of these varieties.

This paper is a direct continuation of the first part appearing with the same title [7]. In the first part, we considered the problem of constructing a (commutative unital) semiring defining the variety of semimodules whose idempotent subreducts lie in a given variety $\mathcal{V}$ of modes, and such that each semimodule-embeddable member of $\mathcal{V}$ embeds into a semimodule over such a semiring. For a given variety $\mathcal{V}$ of modes, such a variety of semimodules was called its semi-linearization, the semiring of the semi-linearization was called the (semi-affinization) semiring of the variety $\mathcal{V}$, and the class of idempotent reducts of the semi-linearization was called the semi-affinization of $\mathcal{V}$. We described the general construction of semi-affinization semirings, with basic examples and some general properties. In the current, second part, we investigate some selected varieties of modes, and provide a description of their semi-affinization semirings. In particular, we investigate varieties of affine spaces, varieties of barycentric algebras, and varieties of semilattice modes. We also show that the semi-affinization of the affinization of a variety of modes is equivalent to the affinization. We provide a new representation theorem for (real and dyadic) barycentric algebras, based on our description

of semi-affinization of the corresponding varieties of such algebras. Finally, we provide a new method of finding the semi-affinization semirings of semi-lattice modes.

Recall, that the Szendrei identity for a binary operation coincides with the entropic law. It follows that all modes with only binary basic operations embed as subreducts into semimodules over commutative semirings.

We use the terminology, notation and results of the first part of the paper, and continue the section numbering from that paper.

4. Semi-affinization of affine spaces

As mentioned in Part I, each variety of Mal'cev modes is equivalent to a variety $\underline{R}$ of affine R-spaces for an appropriate commutative ring R. Moreover, the affinization of $\underline{R}$ coincides with $\underline{R}$ [11], [14, Ch. 4]. One may thus expect that the semi-affinization and the affinization also coincide in such a case, or more precisely, that the semi-affinization of $\underline{R}$ and the variety $\underline{\underline{R}}$ are equivalent varieties. Indeed, we will show that the semiring $S = S(\underline{\underline{R}})$ is just the semiring R, obtained from the ring R by disregarding subtraction, and that the semi-affinization of $\underline{R}$ is a variety equivalent to the variety $\underline{R}$.

Let us begin with the variety $\underline{\mathbb{Z}}$ of integral affine spaces. The variety of such algebras is known to be equivalent to the variety of Mal'cev modes (A, P) with one ternary Mal'cev operation P, denoted also as (xyz). The operations $\underline{n}$ for $n \in \mathbb{Z}$ are defined by means of P [14, Ch. 6]. Recall that in a semiring $(S, +, o, \cdot)$, two elements x and y are said to be *opposite* if $x + y = o$. If there is $p \in S$ such that the unit 1 and p are opposite, then S is a ring. Indeed, for each element $s \in S$, we have

$$s \cdot 1 + s \cdot p = s(1 + p) = s \cdot 0 = 0.$$

Hence the opposite of any $s \in S$ is sp. Moreover, $p^2 = (2p + 1)^2 = 4p(p + 1) + 1 = 1$, and more generally $p^{2n} = 1$ and $p^{2n+1} = p$.

PROPOSITION 4.1. *The semiring $S(\underline{\mathbb{Z}})$ of the variety $\underline{\mathbb{Z}}$ of integral affine spaces is isomorphic to the semiring $\mathbb{Z}$ of integers. The semi-affinization of $\underline{\underline{\mathbb{Z}}}$ is a variety equivalent to the variety of integral affine spaces.*

Proof. The semiring $S(\underline{\mathbb{Z}})$ is calculated as a quotient of the semiring $\mathbb{N}[d, e, f]$. The operation P is defined in each $S(\underline{\mathbb{Z}})$-semimodule by

$$(xyz) = xd + ye + zf.$$

It is idempotent, and satisfies the Mal'cev identities $(xyy) = x = (yyx)$. These identities imply the following relations between d, e, f: we have $d + e + f = 1, d + e = 0$ and $e + f = 0$, and hence also $d = 1 = f$ and $1 + e = 0$. It follows that the opposite of 1 is e and the semiring $S(\underline{\mathbb{Z}})$ is a ring. It is clear now that each element of this ring can be written as $n + ie$

for natural n and i. Hence the ring $S(\underline{\underline{\mathbb{Z}}})$ is (isomorphic to) the ring $\mathbb{N}[j]$ with $j + 1 = 0$, and to the ring $\mathbb{N}[X]/cg(X + 1, 0)$.

The Mal'cev operation P of each $S(\underline{\underline{\mathbb{Z}}})$-semimodule A is defined by

$$(xyz) = x + yj + z.$$

The element j generates a subring isomorphic to $\mathbb{Z}$. Now the semiring $\mathbb{N}[X]$ is free over $\{X\}$, and the mapping $h : \mathbb{N}[X] \to \mathbb{Z}; f(X) \mapsto f(-1)$ is a surjective semiring homomorphism uniquely extending the mapping $X \mapsto -1$. Since the kernel of this homomorphism is the congruence $cg(X + 1, 0)$, it follows that the semirings $\mathbb{N}[X]/cg(X + 1, 0)$ and $\mathbb{Z}$ are isomorphic.

Since the semiring $S(\underline{\underline{\mathbb{Z}}})$ is actually the ring $\mathbb{Z}$, it follows that the linearization and the semi-linearization of $\underline{\underline{\mathbb{Z}}}$ coincide (or more precisely are equivalent). Thus the same holds for the affinization and semi-affinization. ∎

Each subvariety of $\underline{\underline{\mathbb{Z}}}$ has the form $\underline{\underline{\mathbb{Z}_n}}$, where $n \in \mathbb{Z}^+$, and is defined by one additional identity

(4.1) $$(y(x(y \ldots y)x)y) = x,$$

where y is repeated n times.

PROPOSITION 4.2. *For each $n \in \mathbb{N}$, the semiring $S(\underline{\underline{\mathbb{Z}_n}})$ of the variety $\underline{\underline{\mathbb{Z}_n}}$ of affine $\mathbb{Z}_n$-spaces is isomorphic to the semiring $\mathbb{Z}_n$ of integers modulo n. The semi-affinization of $\underline{\underline{\mathbb{Z}_n}}$ is a variety equivalent to the variety $\underline{\underline{\mathbb{Z}_n}}$.*

Proof. For a given integer n, the semiring $S(\underline{\underline{\mathbb{Z}_n}})$ is obtained as a quotient of the semiring $S(\underline{\underline{\mathbb{Z}}})$, determined by the identity (4.1). Since, in integral affine spaces, $(xyz) = x + yj + z$, moreover $j^{2k} = 1$ and $j^{2k+1} = j$, the left-hand side of (4.1) equals $2x(j + j^3 + \cdots + j^{2k-1}) + 2y(1 + j^2 + \cdots + j^{2k-2}) + yj^{2k} = y(2k + 1) + x2kj$ in the case $n = 2k + 1$, and $2x(j + j^3 + \cdots + j^{2k-3}) + 2y(1 + j^2 + \cdots + j^{2k-2}) + xj^{2k-1} = y2k + x(2k - 1)j$ in the case $n = 2k$. Hence $yn + x(n-1)j = x$, which shows that $n = 0$ in $S(\underline{\underline{\mathbb{Z}_n}})$, and finally that $S(\underline{\underline{\mathbb{Z}_n}})$ is isomorphic to the semiring $\mathbb{Z}_n$.

The remaining part is proved similarly as in the previous proposition. ∎

Now, let us consider the general case of affine spaces over an arbitrary commutative unital ring R.

THEOREM 4.3. *The semi-affinization semiring $S(\underline{R})$ of the variety $\underline{R}$ of affine spaces over R is isomorphic to the semiring R. The semi-affinization of $\underline{R}$ is a variety equivalent to the variety $\underline{R}$.*

Proof. As each affine R-space has the Mal'cev operation P, it follows that the semiring $S(\underline{R})$ is a ring and contains the ring $\mathbb{N}[j]$ with $j + 1 = 0$, isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some $n \in \mathbb{N}$, as a subring.

As before, the semiring $S(\underline{R})$ is constructed as the quotient

$$\mathbb{N}[X, X_r, Y_r \mid r \in R]/cg((X+1, 0), (X_r + Y_r, 1) \mid r \in R).$$

The congruence of the free semiring in the latter formula is the kernel of the homomorphism $h : \mathbb{N}[X, X_r, Y_r \mid r \in R] \rightarrow R$ uniquely extending the mapping $X \mapsto -1$. It follows that the semirings $S(\underline{R})$ and R are isomorphic. And hence the semi-affinization of $\underline{R}$ is equivalent to $\underline{\underline{R}}$. ∎

As a corollary one obtains the following.

COROLLARY 4.4. *The semi-affinization of the affinization of a variety of modes is a variety equivalent to the affinization.*

5. Cancellative modes

It is evident that each mode that embeds as a subreduct into an affine space over R (and hence into a module over R) also embeds into a semi-module (the reduct of the module) over a commutative semiring (the reduct of the ring R). However, as we will see, this semimodule is not necessarily minimal.

In [13], Romanowska and Smith showed that each cancellative mode embeds into an affine space. (See also [14, Ch. 7].) Recall that an Ω-mode (C, Ω) of a given type τ is *cancellative* if it satisfies the quasi-identity

$$(x_1 \ldots x_{i-1} y x_{i+1} \ldots x_n \, \omega = x_1 \ldots x_{i-1} z x_{i+1} \ldots x_n \, \omega) \rightarrow (y = z)$$

for each (n-ary) $\omega \in \Omega$ and each $i = 1, \ldots, n$. Since cancellative members of a given variety $\mathcal{V}$ of modes belong to the class of all $\mathcal{V}$-modes embeddable into affine spaces, they all embed into affine spaces over $R(\mathcal{V})$. Note that semilattices are not cancellative, and are not embeddable into affine spaces. On the other hand, both cancellative modes and semilattices embed into semimodules.

Recall that each Ω-mode with a homomorphism onto an Ω-semilattice (an Ω-algebra equivalent to a semilattice) can be represented as so-called semilattice (Lallement) sum. In [13] (see also [14, Ch. 7]), Romanowska and Smith showed that each semilattice sum of cancellative modes (a semilattice sum of cancellative fibres) embeds as a subreduct into a Płonka sum of affine spaces over a ring, say R, common for all these affine spaces. As observed in [15], such a Płonka sum embeds into a Płonka sum of R-modules, and since this Płonka sum of R-modules is a semimodule over R, it follows that each semilattice sum of cancellative modes embeds into a semimodule over a ring. If the cancellative modes in question are all members of a given variety $\mathcal{V}$ of modes, then one may take the ring to be $R(\mathcal{V})$.

Let $\mathcal{SL}_\Omega$ be the variety of Ω-semilattices. Let $\mathcal{CV}$ be the quasivariety of cancellative members of a variety $\mathcal{V}$ of Ω-modes. Then each algebra (A, Ω)

in the quasivariety $\mathcal{CV} \circ \mathcal{SL}_\Omega$, the Mal'cev product relative to the variety of Ω-modes, is known to be a semilattice sum of $\mathcal{CV}$-subalgebras (A_i, Ω) over an Ω-semilattice (I, Ω) [14, Chs. 4, 7]. Hence (A, Ω) embeds as a subreduct into a semimodule over the (semi)ring $R(\mathcal{V})$. Note, however, that a $\mathcal{CV}$-mode also embeds into a semimodule over the semiring $S(\mathcal{V})$. If $\mathcal{V}$ is a regular variety, it contains $\mathcal{SL}_\Omega$ as a subvariety, and Ω-semilattices also embed into $S(\mathcal{V})$-semimodules.

The proof of the following theorem is similar, but slightly easier than, the case of affine space embeddings [14, §7.8].

THEOREM 5.1. *Let $\mathcal{V}$ be a regular variety of Ω-modes. Let a $\mathcal{V}$-mode (A, Ω) be a semilattice sum of cancellative modes (A_i, Ω) over a semilattice (I, Ω). Then (A, Ω) is a subreduct of a Płonka sum of $S(\mathcal{V})$-semimodules over (I, Ω).*

Now a Płonka sum of semimodules over a given semiring is again a semimodule over the same semiring. (Compare [14, §7.8] for the definition of the zero element in the Płonka sum of (semi)modules.) One obtains the following corollary.

COROLLARY 5.2. *Let a $\mathcal{V}$-mode (A, Ω) be a semilattice sum of cancellative modes (A_i, Ω) over a semilattice (I, Ω). Then (A, Ω) embeds as a subreduct into an $S(\mathcal{V})$-semimodule.*

Corollary 5.2 gives a homogeneous embedding and homogenous representation of all members of the quasivariety $\mathcal{CV} \circ \mathcal{SL}_\Omega$ as subreducts of $S(\mathcal{V})$-semimodules. It may be considered as an improvement of the earlier result of Romanowska and Smith. The following two examples provide typical applications of Corollary 5.2. At the same time, they also offer new representations of some barycentric algebras.

EXAMPLE 5.3. Let R be a subfield of the field $\mathbb{R}$ of real numbers. Let R_0^+ be the semiring of non-negative reals in R. As noted in [10, Exs. 6.10, 7.3, 7.7], convex sets over R may be considered not only as subreducts of affine spaces over R, but also as subreducts of R_0^+-semimodules. In fact, the coefficients of any non-trivial idempotent operation of an R_0^+-semimodule belong to the open unit interval I^o of R_0^+. Hence convex sets over R_0^+, considered as algebras $(C, \underline{I}^o)$, are just subalgebras of semi-affine spaces over R_0^+. Now in contrast to the case of the field R, the semiring R_0^+ has a non-trivial congruence relation with $\{0\}$ as a singleton class and a second class consisting of all positive numbers in R. The two-element lattice is the quotient by this congruence. Thus all $\underline{I}^o$-semilattices embed as subreducts of R_0^+-semimodules. The class of convex sets generates the variety $\mathcal{BA}$ of barycentric algebras over R. The class of $\underline{I}^o$-semilattices is the unique non-trivial subvariety of $\mathcal{BA}$.

By results of Romanowska-Smith (see e.g. [14, Th. 7.5.10]), each barycentric algebra $(B, \underline{I}^o)$ is a semilattice sum of open convex sets over its semilattice replica. Convex sets are known to be precisely the cancellative barycentric algebras. Hence, by Theorem 5.1, each barycentric algebra $(B, \underline{I}^o)$ embeds into a Płonka sum of semi-affine R_0^+-spaces over its semilattice replica. By Corollary 5.2, this implies the following theorem.

THEOREM 5.4. *Each barycentric algebra $(B, \underline{I}^o)$ embeds as a subreduct into an R_0^+-semimodule.*

This theorem enhances the Romanowska-Smith theorem that each barycentric algebra embeds as a subreduct into a semimodule that is a Płonka sum of R-modules.

EXAMPLE 5.5. A similar method may be used in the case of "barycentric algebras" over (ordered) subrings of $\mathbb{R}$. In particular, we consider such algebras over the ring $\mathbb{D}$ of rational dyadic numbers $m2^{-n}$ for $m, n \in \mathbb{Z}$. In this case, the set of barycentric operations $\underline{r}$ for r in the open unit interval $\mathbb{D}^o := \mathbb{D} \cap I^o$ is generated by the unique multiplication operation $\underline{1/2}$, so that the algebras in question are equivalent to groupoids. As the multiplication operation is commutative, these groupoids are called commutative binary modes. (See [11, Ch. 4], [14, §5.5], [6].)

A subset of an affine space over $\mathbb{D}$ is called a *dyadic convex set* if it is the intersection of a real convex set with this affine $\mathbb{D}$-space. Let $\mathbb{D}_0^+$ be the semiring of non-negative dyadic numbers. Convex sets over $\mathbb{D}$, as groupoids $(C, \underline{1/2})$, may be considered as convex sets over $\mathbb{D}_0^+$. They are subalgebras of semi-affine spaces over $\mathbb{D}_0^+$.

Let $\mathcal{CBM}$ be the variety of all commutative binary modes. The affinization ring $R(\mathcal{CBM})$ of this variety is the ring $\mathbb{D}$. The semiring $S(\mathcal{CBM})$ of $\mathcal{CBM}$ may be calculated using our general procedure.

LEMMA 5.6. *The semiring $S = S(\mathcal{CBM})$ of the variety $\mathcal{CBM}$ is the semiring $\mathbb{D}_0^+$ of non-negative dyadic numbers.*

Proof. Our general procedure of calculating the semiring of a variety shows that the semiring S is (isomorphic to) the semiring $\mathbb{N}[d, e]$ with $d = e$, and hence with $2d = 1$. We will show that each element $s \in S$ has the form nd^k for some $k, n \in \mathbb{N}$.

First note that for $i, n \in \mathbb{Z}^+$,

(5.1) $$2^i d^n = d^{n-i}.$$

Indeed, for $i = 1$ and any n, this follows directly from the fact that $2d = 1$. If (5.1) holds for some i, then

$$2^{i+1} d^n = 2 \cdot 2^i d^n = 2d^{n-i} = d^{n-i-1}.$$

Hence (5.1) holds for all positive integers.

Now note that for $j = k + i$ and $i \in \mathbb{N}$, one has

$$md^k + nd^j = m2^i d^{k+i} + nd^j = m2^i d^j + nd^j = (m2^i + n)d^j.$$

As each element of S has the form of a polynomial with one indeterminate d and natural coefficients, this implies that each element of S equals nd^k for some natural numbers n and k. Since $S = \mathbb{N}[d]$, where $2d = 1$, it follows that S is isomorphic to $\mathbb{N}[1/2] = \mathbb{D}_0^+$. $\blacksquare$

As in the case of barycentric algebras over a subfield of the field of real numbers, each commutative binary mode is a semilattice sum of cancellative subalgebras [14, Th. 7.5.5]. Again, as in the case of (real) barycentric algebras, one obtains the following theorem.

THEOREM 5.7. *Each commutative binary mode embeds as a subreduct into a $\mathbb{D}_0^+$-semimodule.*

This theorem enhances an earlier result saying that each commutative binary mode embeds as a subreduct into a $\mathbb{D}$-semimodule [14, Cor. 7.8.6].

Unlike real barycentric algebras, commutative binary modes form infinitely many varieties. The lattice $L(\mathcal{CBM})$ of all subvarieties of $\mathcal{CBM}$ was described by Ježek and Kepka [3]. (See also [11, §4.5].) It consists of irregular subvarieties $\mathcal{C}_{2k+1}$, for $k \in \mathbb{N}$, their regularizations $\widetilde{\mathcal{C}_{2k+1}}$, and the variety $\mathcal{CBM}$. Each irregular variety $\mathcal{C}_{2k+1}$ is equivalent to the variety $\underline{\underline{\mathbb{Z}_{2k+1}}}$ of affine spaces. By results of Section 4 and Proposition 2.5 of Part I, it follows that the semi-affinization semiring of the variety $\underline{\underline{\mathbb{Z}_{2k+1}}}$, and also of the variety $\mathcal{C}_{2k+1}$, is just the semiring $\mathbb{Z}_{2k+1}$. By Theorem 3.3 of Part I, the semiring of the regularization $\widetilde{\mathcal{C}_{2k+1}}$ is obtained by adding a new zero to the semiring $\mathbb{Z}_{2k+1}$. This provides the following corollary.

COROLLARY 5.8. *For each natural number k, the semi-affinization semiring $S(\mathcal{C}_{2k+1})$ of the subvariety $\mathcal{C}_{2k+1}$ of $\mathcal{CBM}$ is the semiring $\mathbb{Z}_{2k+1}$. The semiring $S(\widetilde{\mathcal{CBM}})$ of its regularisation is the semiring $\mathbb{Z}_{2k+1} \cup \{z\}$ extending $\mathbb{Z}_{2k+1}$ by adjoining a new zero element z.*

6. Semilattice modes

As considered by Kearnes [4], *semilattice modes* are modes having a semilattice operation. (Such an operation is necessarily unique.) Define a *semilattice semiring* to be a commutative semiring with a semilattice additive reduct satisfying the identity $s + 1 = 1$. Then define a *semilattice semimodule* to be a semimodule, with a semilattice monoid reduct, whose scalar semiring is a semilattice semiring. A consequence of results of [4] is that each semilattice mode embeds as a subreduct into a semilattice semimodule.

Let $\mathcal{V}$ be a variety of semilattice modes. Kearnes constructed a semilattice semiring $S_K(\mathcal{V})$, based on a certain subset of the free $\mathcal{V}$-algebra on two generators, such that each $\mathcal{V}$-algebra embeds as a subreduct into a semilattice semimodule over $S_K(\mathcal{V})$. Kearnes' result relied on a detailed analysis of the structure of subdirectly irreducible semilattice modes. However, using the fact that each such subdirectly irreducible algebra contains a smallest element 0, semilattice modes are readily seen to satisfy Szendrei identities [9]. Consequently, semilattice modes must be subreducts of semimodules over commutative semirings. We will show that the semi-affinization semiring $S(\mathcal{V})$ of $\mathcal{V}$ coincides with $S_K(\mathcal{V})$.

PROPOSITION 6.1. *Let $\mathcal{V}$ be a variety of semilattice modes. Then the semiring $S(\mathcal{V})$ is a semilattice semiring.*

Proof. Without loss of generality, assume that the semilattice operation of a $\mathcal{V}$-mode is a basic operation. Let $\mathcal{V}_\tau$ be the variety of all semilattice modes $(A, +, \Omega)$ of a fixed finite type $\tau : \Omega \to \mathbb{N}$.

The semi-affinization semiring $S(\mathcal{V}_\tau)$ is calculated using our general procedure as the quotient of the semiring $\mathbb{N}[X, Y, X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega\tau]$ over a set of commuting indeterminates by the congruence

$$\theta = cg\Big((X + Y, 1), (X, Y), \Big(\sum_{i=1}^{\omega\tau} X_{\omega i}, 1\Big) \,\Big|\, \omega \in \Omega\Big).$$

In particular, in the semiring $S(\mathcal{V}_\tau)$, we have $X = Y = 1$. The additive reduct of the semiring $S(\mathcal{V}_\tau)$ (and of each $S(\mathcal{V}_\tau)$-semimodule) is obviously a semilattice. We will show that the identity $s + 1 = 1$ is satisfied in $S(\mathcal{V}_\tau)$.

Denote by $a_{i,\omega}$ the congruence class $(X_{\omega i})^\theta$, where the $X_{\omega i}$ pertain to $\omega \in \Omega$ and $1 \leq i \leq n_\omega = \omega\tau$. Let A be the set of all $a_{i,\omega}$ for $\omega \in \Omega$. Note that $n \in 1^\theta$ for each positive integer n. Similarly $n a_{i_1,\omega_1}^{k_1} \ldots a_{i_r,\omega_r}^{k_r} = a_{i_1,\omega_1}^{k_1} \ldots a_{i_r,\omega_r}^{k_r}$ for any natural numbers $k_1, \ldots, k_r$. It follows that each representative of $S(\mathcal{V}_\tau)$ may be written in the form of a polynomial with the variables in A and with coefficients equal to 0 or 1. Now for any $\omega \in \Omega$ and $i = 1, \ldots, n = n_\omega$,

$$1 + a_{i,\omega} = a_{1,\omega} + \cdots + a_{n,\omega} + a_{i,\omega} = a_{1,\omega} + \cdots + a_{n,\omega} = 1.$$

If $1 + a_{i,\omega}^r = 1$, then similarly as above

$$1 + a_{i,\omega}^{r+1} = a_{1,\omega} + \cdots + a_{n,\omega} + a_{i,\omega}^{r+1}$$

$$= a_{1,\omega} + \cdots + a_{i,\omega}(1 + a_{i,\omega}^r) + \cdots + a_{n,\omega} = a_{1,\omega} + \cdots + a_{n,\omega} = 1.$$

A similar induction proof shows more generally that

$$1 + a_{i_1,\omega_1}^{k_1} \cdot \ldots \cdot a_{i_r,\omega_r}^{k_r} = 1,$$

for any natural numbers $k_1, \ldots, k_r$. Consequently, adding 1 to any element of $S(\mathcal{V}_\tau)$ will give 1, i.e for each $s \in S$, we have $1 + s = 1$.

Finally, if $\mathcal{V}$ is a subvariety of $\mathcal{V}_\tau$, then the semi-affinization semiring $S(\mathcal{V})$ is a homomorphic image of $S(\mathcal{V}_\tau)$, whence it also satisfies the required properties. ∎

EXAMPLE 6.2. In particular, any mode in the variety $\mathcal{V}$ of modes with a semilattice operation $+$ and one binary operation $\cdot$ embeds into a semimodule. The semiring $S(\mathcal{V})$ is calculated as in Proposition 6.1. As in [4, Exercise], one shows that $a_1 + a_2 = 1$ implies that $a_1^i + a_2^j = 1$, whence the elements of $S(\mathcal{V})$ may be represented as 0 or $a_1^i \cdot a_2^j$ for $i, j \geq 0$. This shows that the semiring $S(\mathcal{V})$ coincides with the semiring calculated in [4, Exercise].

THEOREM 6.3. *Suppose that $\mathcal{V}$ is a variety of semilattice modes of type $\tau : \Omega \cup \{+\} \to \mathbb{N}$. Then the semirings $S(\mathcal{V})$ and $S_K(\mathcal{V})$ coincide.*

Proof. By [4, §4.2], the semiring $S_K(\mathcal{V})$ is built on the interval $[y, x+y]$ of the free $\mathcal{V}$-mode $F_\mathcal{V}(x, y)$ over the set $\{x, y\}$, and consists of all terms $t(x, y)$ such that $t(x, y) = t(x, y) + y = t(x + y, y)$. On the other hand, by [1, §V.1], the free $S(\mathcal{V})$-semimodule over $\{x, y\}$ is isomorphic to the direct product $S(\mathcal{V}) \times S(\mathcal{V})$ with free generators $x = (1, 0)$ and $y = (0, 1)$. Recall also (compare Pt. I, §2) that the free Szendrei Ω-mode on set A is isomorphic to the Ω-subreduct generated by A of the free $S(\tau)$-semimodule over A. It follows that the free $\mathcal{V}$-mode $F_\mathcal{V}(x, y)$ is the subreduct of the $S(\mathcal{V})$-semimodule $S(\mathcal{V}) \times S(\mathcal{V})$ generated by x and y. It is clear that the interval $[y, x + y]$ of $S(\mathcal{V}) \times S(\mathcal{V})$ is (isomorphic to) a subalgebra of $F_\mathcal{V}(x, y)$, and is isomorphic to the $\mathcal{V}$-reduct of $S(\mathcal{V})$. It is also easy to see that, as a semiring, $S(\mathcal{V})$ is isomorphic to $S_K(\mathcal{V})$. ∎

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