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DYADIC POLYGONS*

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Dyadic rationals are rationals whose denominator is a power of 2. *Dyadic triangles* and *dyadic polygons* are, respectively, defined as the intersections with the dyadic plane of a triangle or polygon in the real plane whose vertices lie in the dyadic plane. The one-dimensional analogs are *dyadic intervals*. Algebraically, dyadic polygons carry the structure of a commutative, entropic and idempotent algebra under the binary operation of arithmetic mean. In this paper, dyadic intervals and triangles are classified to within affine or algebraic isomorphism, and dyadic polygons are shown to be finitely generated as algebras. The auxiliary results include a form of Pythagoras' theorem for dyadic affine geometry.

Keywords: Affine space; convex set; polygon; polytope; commutative; binary; entropic mode; arithmetic mean.

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1. Introduction

The mathematical objects considered in this paper share geometric and algebraic aspects. As geometric objects, they are subsets of certain affine spaces. As algebraic objects, they have one binary idempotent and entropic operation.

Consider the ring $\mathbb{D} = \mathbb{Z}[1/2]$ of dyadic rational numbers, numbers of the form $m2^{-n}$ for $m, n \in \mathbb{Z}$. The affine spaces of interest are affine spaces over $\mathbb{D}$. They are subreducts (subalgebras of reducts) of affine spaces over the ring $\mathbb{R}$ of reals. The geometric objects we consider are certain convex subsets of dyadic affine spaces. A subset of $\mathbb{D}^k$ is called a *dyadic convex set* if it is the intersection of a real convex set with the space $\mathbb{D}^k$. Such sets may also be called *convex relative to* $\mathbb{D}$ (cf. for instance, [1]). *Dyadic polytopes* are dyadic convex sets obtained as the intersection of real polytopes and dyadic spaces whose vertices lie in the dyadic space. *Dyadic triangles* and *dyadic polygons* are, respectively, defined as the intersections with the dyadic plane of a triangle or polygon in the real plane whose vertices lie in the dyadic plane. The one-dimensional analogs are *dyadic intervals*.

While dyadic geometry forms a fascinating field of study from the purely mathematical standpoint, there are also other reasons for which it may be of interest. Since its points are coordinatized by finite binary expressions, it is the precise geometry that is represented in a digital computer. On the other hand, it provides a good model for a space with holes that may be relevant to certain areas of high-energy physics.

Real convex sets are described algebraically as certain barycentric algebras. In particular, they can be characterized as subsets of some $\mathbb{R}^k$ closed under the operations of weighted means coming from the open real unit interval I°. Thus a convex set contains, along with any two of its points, the line segment joining them. Similarly, dyadic convex sets may be characterized as subsets of some $\mathbb{D}^k$ which contain, along with any two of their points, the dyadic line segment joining them. As barycentric algebras, all such real segments are isomorphic to the closed unit interval, and are generated by their endpoints. In contrast, the dyadic segments are not necessarily generated by their endpoints. However, as we will see later in this paper, all their elements can be obtained from a two- or three-element set of generators using only one idempotent, commutative and entropic operation

$$x \cdot y := xy\underline{1/2} = \frac{x+y}{2}.$$

This endows each dyadic convex set with the algebraic structure of a commutative binary mode ($\mathcal{CB}$-mode for short), or in other words an idempotent, commutative and entropic groupoid. Such algebras have a well-developed algebraic theory [3–7, 11–13]. In many aspects, commutative binary modes behave like barycentric algebras (see [11, 13]). However, there are also essential differences. For example, real convex sets are characterized as cancellative barycentric algebras, whereas not all cancellatice $\mathcal{CB}$-modes are convex sets [8]. As we will see, not all properties of real polytopes carry over to their dyadic counterparts.

Our interest in the geometric aspects of $\mathcal{CB}$-modes was stimulated by the intriguing differences between real and dyadic convexity, and the generation problem for dyadic polytopes that arose from [8, 9].

Problem 1.1. Are all dyadic polytopes finitely generated?

The current paper initiates investigations of dyadic convex sets as commutative binary modes. In particular, we are interested in dyadic (closed) intervals and dyadic polygons. Following some preliminaries collected in Sec. 2, Sec. 3 shows that up to isomorphism, there are infinitely many dyadic intervals. This clearly contrasts with the real case. We provide a full classification and show that, with one exception, they are generated by no less than three generators. These results are then used in Sec. 4 to investigate certain right triangles that play a basic role in the later work. In Sec. 5, the second main result is presented. We show that each dyadic triangle belongs to one of three basic types of triangles, and provide certain characterizations of these three types. In affine geometry, all real triangles are isomorphic. But there are infinitely many isomorphism types of dyadic triangle. These types are classified in Sec. 6 by a complete invariant, the encoding quadruple. In Sec. 7, we show that each dyadic triangle is finitely generated, and conclude that all dyadic polygons are finitely generated.

Although this paper should be mostly self-contained, we refer readers to our monographs [11, 13, 14] for additional information about algebraic concepts used in the paper. Our notation generally follows the conventions established there.

2. Preliminaries

Most algebras considered in this paper are *modes* in the sense of [11, 13], algebras in which each element forms a singleton subalgebra, and for which each operation is a homomorphism. One of the main classes of modes is given by affine spaces over commutative, unital rings R (affine R-spaces), or more generally, by subreducts (subalgebras of reducts) of affine spaces. Affine spaces are considered here as Mal'cev modes, as explained in the monographs [11, 13]. In particular, if R is the ring $\mathbb{R}$ or the ring $\mathbb{D}$, affine R-spaces can be considered as the reducts $(A, \underline{R})$ of R-modules $(A, +, R)$, where $\underline{R}$ is the family of binary operations

$$\underline{r} : A^2 \to A; \quad (x_1, x_2) \mapsto x_1 x_2 \underline{r} = x_1(1 - r) + x_2 r$$

for each $r \in R$. The class of all affine R-spaces forms a variety (cf. [2]).

The focus in this paper falls on one- and two-dimensional dyadic and real spaces. A dyadic space of a given dimension will be considered as a subspace of the corresponding real space, and may be equipped with the usual coordinate axes. These spaces will be considered as $\mathcal{CB}$-modes, and as corresponding reducts of affine spaces. Similarly, dyadic polytopes will be considered as $\mathcal{CB}$-modes, and as corresponding subreducts of dyadic spaces.

Real polytopes are defined abstractly as finitely generated convex sets, with vertices as generators. As we will see, not all of the properties characteristic of real polytopes carry over to the dyadic case.

Recall first that a *wall* W of a groupoid $(A, \cdot)$ is a subset of A such that $a \cdot b \in W$ precisely when $a \in W$ and $b \in W$. (See [13] for a more general definition, and a geometric interpretation.) The vertices of a dyadic polytope are its only one-element walls. However, it will transpire that the vertices do not necessarily suffice as generators. Examples of dyadic polytopes generated by their vertices are provided by finitely generated free $\mathcal{CB}$-modes. Recall that a free commutative binary mode $X\mathcal{CB}$ on a finite set $X = \{x_0, x_1, \ldots, x_n\}$ is isomorphic to the $\underline{2^{-1}}$-reduct generated by X of the free affine $\mathbb{D}$-space $X\mathbb{D}$ on X (see [10, Lemma 6.1]). The set of its elements can be identified with the set

$$\left\{x_0a_0 + \cdots + x_na_n \,\middle|\, a_i \in \mathbb{D}_1, \sum a_i = 1\right\},$$

where $\mathbb{D}_1$ denotes the *dyadic unit interval* $\mathbb{D} \cap [0, 1]$. Thus $X\mathcal{CB}$ forms an n-dimensional *dyadic simplex* S_n (cf. [5]). Note that S_1 coincides with $\mathbb{D}_1$.

Automorphisms of the affine dyadic space $\mathbb{D}^n$ form the n-dimensional affine group $\mathrm{GA}(n, \mathbb{D})$ over the ring $\mathbb{D}$, the group generated by the linear group $\mathrm{GL}(n, \mathbb{D})$ over $\mathbb{D}$ and the group of translations of the space $\mathbb{D}^n$. The following observation will frequently be used later on.

Proposition 2.1. *Each element of the affine group* $\mathrm{GA}(n, \mathbb{D})$ *is also an automorphism of the groupoid* $(\mathbb{D}^n, \circ)$, *where* $\circ = \underline{1/2}$. *Moreover, it transforms any polytope in* $\mathbb{D}^n$ *into an isomorphic polytope.*

Proof. It suffices to note that each automorphism of an algebra is also an automorphism of each reduct, and that automorphisms of an algebra transform subalgebras to isomorphic subalgebras. □

On the other hand, each non-trivial n-dimensional polytope contains an n-dimensional simplex as a subalgebra. As such a simplex is a subreduct of the free dyadic affine space on the same set of generators, each automorphism of the polytope extends to an automorphism of the affine space. We will use this observation to consider isomorphisms of polytopes in $\mathbb{D}^n$ as restrictions of automorphisms of the affine space $\mathbb{D}^n$.

Now note that for $n = 1$, the affine automorphisms of $\mathbb{D}$ are given by

$$\kappa(d, a) : \mathbb{D} \to \mathbb{D}; \quad x \mapsto dx + a,$$

where $d = 2^r$ for some integer r, and $a \in \mathbb{D}$. The linear group $\mathrm{GL}(1, \mathbb{D})$ is isomorphic to the group of units $\mathbb{D}^*$, consisting of the elements $d = 2^r$ for integral r.

If $n = 2$, then the linear group $\mathrm{GL}(2, \mathbb{D})$ consists of invertible dyadic 2×2 matrices with determinants equal to 2^r for some integer r. Note that a dyadic matrix

$$M = \begin{bmatrix} a2^i & b2^j \\ c2^k & d2^l \end{bmatrix},$$

where a, b, c and d are odd integers, is a member of $\mathrm{GL}(2, \mathbb{D})$ precisely when $\gcd\{ad, cb\} = 1$.

As a final remark, let us note that any dyadic n-gon with vertices $A_1, \ldots, A_n$ may be decomposed as a union of triangles

$$A_1A_2A_3,\ A_1A_3A_4, \ldots, A_1A_{n-1}A_n,$$

much as in the case of real n-gons. So to understand the structure and properties of dyadic n-gons, it is first essential to understand the structure and properties of dyadic intervals and dyadic triangles.

3. Dyadic Intervals

This section will classify the dyadic intervals up to isomorphism. In contrast with the real case, it transpires that there are infinitely many pairwise non-isomorphic dyadic intervals. We say that a dyadic interval is *m-generated* if it can be generated by m, but no fewer, elements. With the exception of intervals isomorphic to the one-dimensional dyadic simplex $\mathbb{D}_1$ or S_1, each dyadic interval is 3-generated. We begin with the following observation.

Proposition 3.1. *Suppose that $k > 1$ is a positive odd integer. Then the interval $[0, k]$ is 3-generated.*

Proof. Let $n = \lfloor \log_2 k \rfloor$, so that 2^n is the largest power of 2 that does not exceed k. The numbers 0 and 2^n generate the full dyadic interval $[0, 2^n]$, including the positive integer $k - 2^n$. Now $[k - 2^n, k]$ is isomorphic to $[0, 2^n]$ and its ends generate all the elements of $[k - 2^n, k]$. It follows that the interval $[0, k] = [0, 2^n] \cup [k - 2^n, k]$ is generated by the three numbers 0, 2^n and k.

Now we show that the interval $[0, k]$ is not generated by less than three elements. Certainly, a generating set must include the endpoints 0 and k. These endpoints generate the set

$$S = \{km/2^r \mid r \in \mathbb{N}, m = 0, 1, \ldots, 2^r\}, \tag{3.1}$$

which does not contain any integer lying strictly between 0 and k. Thus the minimal number of generators is 3. □

It transpires that the intervals considered in Proposition 3.1 actually represent all intervals not isomorphic to the dyadic unit interval $\mathbb{D}_1$. First note that each dyadic interval translates isomorphically to an interval of the form $[0, d\,]$, where d is a positive dyadic number.

Lemma 3.2. *The following hold for dyadic intervals*:

(1) *An interval is generated by its endpoints precisely when it is isomorphic to $\mathbb{D}_1$.*

(2) *For each positive odd integer* k, *and each integer* r, *the intervals* $[0, k]$ *and* $[0, k2^r]$ *are isomorphic.*
(3) *Two intervals* $[0, k]$ *and* $[0, l]$, *where* k *and* l *are odd positive integers, are isomorphic precisely if* $k = l$.

Proof. Assume that an interval $[0, k]$ is generated by its endpoints. To show the first statement, it is sufficient to note that $[0, k]$ is of the form (3.1), and that the mapping

$$h : [0, 1] \to [0, k]; \quad m/2^r \mapsto km/2^r$$

is a groupoid isomorphism. The second and third assertions follow by the fact that two intervals starting at 0 are isomorphic precisely when one is obtained from the other by an action of the linear group $\mathrm{GL}(1, \mathbb{D})$. Now $\mathbb{D}$ is the disjoint union of the $\mathrm{GL}(1, \mathbb{D})$-orbits $\{0\} = 0\mathbb{D}^*$ and $(2n + 1)\mathbb{D}^*$ for non-negative integers n. □

The isomorphism classes of dyadic intervals are determined by the orbits of $\mathrm{GL}(1, \mathbb{D})$ on the set of non-zero dyadic numbers. For an odd positive integer k, the class containing the interval $[0, k]$ is

$$\{[d, d + k2^r] \mid d \in \mathbb{D}, r \in \mathbb{Z}, 0 < k \in 2\mathbb{Z} + 1\}.$$

Theorem 3.3. *Each interval of* $\mathbb{D}$ *is isomorphic to some interval* $[0, k]$, *where* k *is an odd positive integer. Two such intervals are isomorphic precisely when their right-hand ends are equal.*

Definition 3.4. If an interval of $\mathbb{D}$ is isomorphic to some interval $[0, k]$, where k is an odd positive integer, then we will say that it is of *type* k.

Corollary 3.5. *Each dyadic interval of type* $k > 1$ *is minimally generated by three elements.*

4. Dyadic Triangles and Their Boundary Types

By convention, the triangles to be considered in this and subsequent sections are closed dyadic triangles which do not reduce to intervals or points. Such triangles have three sides, each of which forms a dyadic interval. As such, it is classified by the techniques of the preceding section.

Definition 4.1. If the sides of a (dyadic) triangle have respective types m, n and k, then we say that the triangle has *boundary type* (m, n, k).

Note that the type is only defined up to cyclic order. This section will discuss the problems of existence and isomorphism for triangles of a given boundary type, especially for certain right triangles which play a fundamental role in the following two sections.

Recall that automorphisms of the dyadic plane $\mathbb{D}^2$ are described as elements of the affine group $\mathrm{GA}(2, \mathbb{D})$. Among the automorphisms of $\mathbb{D}^2$ that we will use frequently, there are in particular the linear maps given by the matrices

$$\begin{bmatrix} 2^k & 0 \\ a & 2^l \end{bmatrix}, \quad \begin{bmatrix} 2^k & b \\ 0 & 2^l \end{bmatrix}, \tag{4.1}$$

and those given by matrices

$$\begin{bmatrix} a & 2^k \\ 2^l & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2^k \\ 2^l & b \end{bmatrix} \tag{4.2}$$

for any integers k and l and dyadic numbers a and b. Each of these automorphisms transforms any of the triangles in the plane $\mathbb{D}^2$ into an isomorphic triangle.

4.1. *Uniqueness*

Is it possible for non-isomorphic triangles to have the same boundary type? In addressing this question, we consider certain right triangles that are of fundamental importance throughout this work.

Definition 4.2. Consider a dyadic right triangle.

(a) The triangle is said to be of *right type* if its shorter sides lie parallel to the coordinate axes.
(b) Let m and n be odd positive integers. Then the triangle is said to be of *right type* $r_{m,n}$ if its shorter sides lie parallel to the coordinate axes, and have respective interval types m and n.

A representative example of a triangle of right type $r_{m,n}$ is given by the following triangle $T_{m,n}$. It is constructed as a right triangle ABC with the origin as vertex A, the second vertex $B = (m, 0)$ on the x-axis and the third vertex $C = (0, n)$ on the y-axis. Note that

$$AB = \{(l/2^r, 0) \mid 0 \leq l \leq m2^r,\ r \in \mathbb{Z}^+\},$$

and similarly

$$AC = \{(0, l/2^r) \mid 0 \leq l \leq n2^r,\ r \in \mathbb{Z}^+\}.$$

The following result may be regarded as an analog of Pythagoras' theorem in dyadic affine geometry, since it specifies the isomorphism type of the hypotenuse of a right triangle in terms of the isomorphism types of the other two sides. (The classical Pythagorean theorem does the same for Euclidean geometry, where the length of a finite closed interval determines its isomorphism type.)

Theorem 4.3 (Pythagoras' theorem). *Let m and n be given positive odd integers. Then the hypotenuse of a triangle of right type $r_{m,n}$ is of type $s = \gcd\{m, n\}$.*

Proof. As before, let $A = (0,0), B = (m,0)$ and $C = (0,n)$. The vertices B and C belong to the (real) line L given by $y = -(n/m)x + n$. Write $m = m's$ and $n = n's$. The real line L then takes the form $y = -(n'/m')x + n$. The intersection of the real line L with the dyadic line segment BC consists of the points

$$\left(\frac{m'k}{2^r}, n'\left(s - \frac{k}{2^r}\right)\right) \tag{4.3}$$

for natural numbers r, with $0 \leq k/2^r \leq s$. Indeed, if $x = l/2^r$ with $0 \leq l \leq 2^r m$, then $y = -(n'/m')(l/2^r) + n$ lies in $\mathbb{D}$ precisely when $l = m'k$ and $0 \leq k \leq 2^r s$.

Consider the mapping

$$\iota : [0,s] \to CB; \quad z \mapsto (m'z, -n'z + n)$$

from the dyadic interval $[0,s]$ to the dyadic segment CB. By (4.3), ι bijects. Furthermore, ι is a homomorphism with respect to all the operations $\underline{d}$ for $d \in \mathbb{D}_1$. Indeed, for $z_1, z_2 \in [0,s]$ and $d \in \mathbb{D}_1^\circ$, one has

$$\begin{aligned} z_1 z_2 \underline{d}^{\,\iota} &= (z_1(1-d) + z_2 d)^\iota \\ &= (m'(z_1(1-d) + z_2 d), -n'(z_1(1-d) + z_2 d) + n)\,, \end{aligned}$$

and so

$$\begin{aligned} z_1^\iota z_2^\iota \underline{d} &= (m'z_1, -n'z_1 + n)(m'z_2, -n'z_2 + n)\underline{d} \\ &= (m'z_1(1-d) + m'z_2 d, (-n'z_1 + n)(1-d) + (-n'z_2 + n)d) \\ &= (m'(z_1(1-d) + z_2 d), -n'(z_1(1-d) + z_2 d) + n) = (z_1 z_2 \underline{d})^\iota. \end{aligned}$$

Thus $\iota : [0,s] \to CB$ is the desired isomorphism. □

For a given positive odd integer s, there are infinitely many pairwise non-isomorphic right triangles in $\mathbb{D}^2$ with hypotenuse of type s. In fact, for each pair (m,n) of relatively prime natural numbers, there is a right triangle with shorter sides isomorphic to the intervals $[0,ms]$ and $[0,ns]$.

For the triangle $T_{m,n}$ with vertices $A = (0,0)$, $B = (m,0)$ and $C = (0,n)$, let $\tilde{T}_{m,n}$ denote the triangle with vertices $\tilde{C} = (0,0)$, $\tilde{B} = (m,n)$ and $\tilde{A} = (0,n)$. Let $\bar{T}_{m,n}$ be the triangle with vertices $\bar{C} = (m,0)$, $\bar{B} = (0,n)$ and $\bar{A} = (m,n)$. Let $T'_{m,n}$ be the triangle with vertices $C' = (m,n)$, $B' = (0,0)$ and $A' = (m,0)$. Note that each triangle of right type $r_{m,n}$ lies in an affine group orbit represented by each kind of triangle described in the following lemma.

Lemma 4.4. *Let m and n be positive odd natural numbers.*

(a) *Each triangle $T_{m,n}$ is isomorphic with the triangle $T_{n,m}$.*
(b) *The triangles $T_{m,n}$, $\tilde{T}_{m,n}$, $\bar{T}_{m,n}$ and $T'_{m,n}$ are isomorphic.*

Proof. The following matrix gives the isomorphism required for (a):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.4}$$

The first isomorphism required for (b) is given by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.5}$$

and a translation, the second by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.6}$$

and a translation, and the third by (4.5) and a translation. □

Corollary 4.5. *A dyadic triangle of right type is determined uniquely up to isomorphism by its boundary type.*

Example 4.6. Note that there are triangles in $\mathbb{D}^2$ that are not isomorphic to right triangles. An example is given by the triangle with vertices $A = (0, 0)$, $B = (0, 27)$ and $C = (105, 90)$. Its sides are isomorphic with the intervals $[0, 27]$, $[0, 21]$ and $[0, 15]$. Since none of the numbers $15, 21, 27$ are the greatest common divisor of the remaining pair, there is no right triangle isomorphic to the given one.

Theorem 4.3 shows that the triangle $T_{1,1}$, isomorphic to S_2, has boundary type $(1, 1, 1)$. Along with $T_{1,1}$, the triangles with vertices $A = (-2^k, 0), B = (2^k, 0)$ and $C = (0, 2^l)$, for some integers k and l, are isomorphic to S_2. Each such triangle is the union of two right triangles ACC' and $CC'B$ with $C' = (0, 0)$, and each of these is isomorphic to S_2. As C' is generated by A and B, it follows that each point of the triangle ABC is generated by its vertices. Further examples of triangles isomorphic to S_2 are obtained as images of the previous ones under the automorphisms given by the matrices (4.1) and (4.2). Clearly, all have type $(1, 1, 1)$. The following proposition shows that there are triangles of boundary type $(1, 1, 1)$ that are not isomorphic to S_2.

Proposition 4.7. *There are infinitely many pairwise non-isomorphic triangles of boundary type* $(1, 1, 1)$.

Proof. Consider the triangle with vertices $A = (-1, 0)$, $B = (1, 0)$ and $C = (0, n)$, where $n \neq 1$ is an odd positive integer. Set $C' = (0, 0)$. The side AB is of type 1. Since both $AC'C$ and $BC'C$ are right triangles, Theorem 4.3 implies that AC and BC are also of type 1. It follows that the triangle ABC has boundary type $(1, 1, 1)$.

Assume, on the contrary, that the triangle ABC is generated by its vertices. This would mean that the triangle is isomorphic with S_2. As each point of S_2 is a combination of vertices as given in Sec. 2, it follows that there are $a_0, a_1, a_2 \in \mathbb{D}_1$ with $a_0 + a_1 + a_2 = 1$ such that

$$(-1, 0)a_0 + (1, 0)a_1 + (0, n)a_2 = (0, k)\,,$$

where $k = 1, 2, \ldots, n-1$. The only real solution of this equation is $a_0 = a_1 = (n-k)/2n$ and $a_2 = k/n$. However, these numbers are not dyadic. It follows that none of the points $(0, k)$ is generated by the vertices of ABC. Hence to generate ABC, we also need at least one additional generator.

Now consider the triangle with vertices $A = (-1, 0)$, $B = (1, 0)$ and $D = (0, m)$, where m is an odd positive integer different from 1 and n. Since the line segments CC' and DC' have different types, it follows that the triangles ABC and ABD cannot be isomorphic. □

In particular, the triangles ABC that were constructed in the proof of Proposition 4.7 are determined by the type of CC'.

4.2. *Existence*

Given odd positive integers m, n and k, is there at least one triangle of boundary type (m, n, k)?

Proposition 4.8. *Let m, n, k be odd positive integers. If*

$$\gcd\{m, n\} = \gcd\{m, k\} = \gcd\{n, k\}, \tag{4.7}$$

then there is a dyadic triangle of boundary type (m, n, k).

Proof. Set $\gcd\{m, n\} = \gcd\{m, k\} = \gcd\{n, k\} =: d$. Note that $\gcd\{m, n, k\} = d$. Let $m = m'd$, $n = n'd$ and $k = k'd$. Then $\gcd\{m', n'\} = \gcd\{m', k'\} = \gcd\{n', k'\} = 1$. In particular, there are integers l' and q' such that $l'n' + q'k' = 1$. Hence $m' = m'l'n' + m'q'k'$ and

$$m = m'd = m'l'n + m'q'k = ln + qk\,, \tag{4.8}$$

where $l = m'l'$ and $q = m'q'$. Now let us consider the triangle in the plane $\mathbb{D}^2$ with vertices $A = (0, 0)$, $B = (m, 0)$ and $C = (nl, kn/d)$. Let $C' = (nl, 0)$. Then the line segment AC' is isomorphic to the interval $[0, nl]$, and the line segment CC' is isomorphic to the interval $[0, kn/d]$.

Applying Theorem 4.3 to the right triangle $AC'C$, one concludes from (4.8) that the line segment AC is isomorphic to the interval $[0, \gcd(nl, kn/d)] = [0, n]$. Similarly, applying Theorem 4.3 to the right triangle $BC'C$, one concludes that the line segment BC is isomorphic to the interval $[0, \gcd(m \pm nl, kn/d)] = [0, k]$. It follows that there is a dyadic triangle whose sides are isomorphic to the intervals $[0, m]$, $[0, n]$ and $[0, k]$. □

In particular, it follows that, for each positive odd integer m, there is a (right) triangle of boundary type (m, m, m). Similarly, there are triangles of type $(1, 1, m)$.

Other examples are given by triangles of type (m, n, k), where m, n and k are pairwise relatively prime. In this case, however, a right triangle is obtained only for type $(1, m, 1)$. Note that the triangle in $\mathbb{D}^3$ with vertices $(mk, 0, 0)$, $(0, mn, 0)$ and $(0, 0, nk)$ is also of type (m, n, k).

Proposition 4.9. *Let T be a triangle in $\mathbb{D}^2$ of boundary type (m, n, k) with vertices $A = (0, 0)$, $B = (m, 0)$ and $C = (x, y)$ for $x, y > 0$. Then m, n and k satisfy the condition (4.7).*

Proof. Let $C' = (x, 0)$. Theorem 4.3 applied to the right triangles ACC' and BCC' shows that x and y have to satisfy the following equalities

$$y = nj = kp \quad \text{and} \quad x = nl \tag{4.9}$$

for some positive integers j, l, p, q such that $\gcd\{j, l\} = 1 = \gcd\{p, q\}$, where

$$m = nl + kq \quad \text{or} \quad m = nl - kq.$$

Note that each of the last two equations has an integral solution with respect to l and q precisely when $\gcd\{n, k\} \mid m$. Let $g = \gcd\{n, k\}$, and let $m = m'g$, $n = n'g$ and $k = k'g$. In particular $\gcd\{n', k'\} = 1$. Suppose, on the contrary, that the condition (4.7), or the equivalent condition

$$\gcd\{m', n'\} = \gcd\{m', k'\} = \gcd\{n', k'\} = 1\,, \tag{4.10}$$

is not satisfied. Without loss of generality assume that $\gcd(m', n') = h \neq 1$. Note that $\gcd\{n', k'\} = 1$ implies that h does not divide k'. Since $m' - n'l = \pm k'q$, it follows that $h \mid q$. As $\gcd\{p, q\} = 1$, it follows that h does not divide p. Now by (4.9), $n'j = k'p$. But h does not divide $k'p$ and divides $n'j$, which gives a contradiction. It follows that the condition (4.10), and hence (4.7), must be satisfied. □

We are left with the following basic question: is each triangle in $\mathbb{D}^2$ isomorphic to a triangle with one vertex at the origin and one of the remaining two on a coordinate axis? In contrast to the real case, the answer is negative.

Definition 4.10. Consider a point $A = (p2^q, u2^v)$ of $\mathbb{D}^2$, where p, u, q and v are integers, with p and u being odd. Such a point is said to be *axial* if

$$\gcd\{p, u\} \in \{p, u, 1\}\,. \tag{4.11}$$

Lemma 4.11. *Let A be a point of $\mathbb{D}^2$ not on any of the coordinate axes. A $\mathbb{D}$-module automorphism of the plane $\mathbb{D}^2$ transforms A into a point on one of the axes if and only if A is axial.*

Proof. Let $A = (p2^q, u2^v)$, with p, q, u, v as in Definition 4.10. First note that A is transformed into a point of one of the axes precisely when there is an invertible matrix

$$M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

in $\mathrm{GL}(2, \mathbb{D})$ such that the equation

$$(p2^q, u2^v)M = (w, 0)\,,$$

or a similar equation with $(0, w)$ in place of $(w, 0)$, holds for $w \neq 0$. Without loss of generality, we may just consider the first case. Also, we may assume that the determinant of M equals 2^r for some integer r. This means that the system of equations

$$xt - yz = 2^r, \tag{4.12}$$

$$p2^q x + u2^v z - w = 0, \tag{4.13}$$

$$p2^q y + u2^v t = 0 \tag{4.14}$$

has a dyadic solution. So we want to show that this system has a solution if and only if (4.11) holds.

Note that it is impossible for a column or row of M to consist entirely of zeros. Nor it is possible for y or t to be zero, since in such a case the second variable would also be equal to zero.

Assume that the system has a solution, and that one of x and z, say z, is zero. In this case $x = \pm 2^k$ and $t = \pm 2^l$ for some integers k and l. Equations (4.13) and (4.14) take the form

$$\pm p2^{q+k} = w, \tag{4.15}$$

$$p2^q y \pm u2^{v+l} = 0. \tag{4.16}$$

The second equation has a dyadic solution precisely when p divides u. In similar fashion, one shows that if $x = 0$, then u divides p.

Now let us consider the case when all of x, y, z, t differ from 0. Equation (4.14) has a solution

$$t = p2^q \quad \text{and} \quad y = -u2^v. \tag{4.17}$$

By subtracting Eq. (4.14) multiplied by z from (4.13) multiplied by t, and taking (4.12) into account, one obtains

$$p2^{q+r} = wp2^q,$$

whence

$$w = 2^r,$$

and Eq. (4.13) takes the form

$$p2^q x + u2^v z = 2^r. \tag{4.18}$$

Now it is easy to see that Eq. (4.18) is equivalent to the equation

$$p2^a x + u2^b z = 2^c \tag{4.19}$$

for some non-negative integers a, b and c. Equation (4.19) has an integral solution precisely when $\gcd\{p2^a, u2^b\}$ divides 2^c. As both p and u are odd integers, this happens precisely when $\gcd\{p, u\} = 1$ and $\min\{a, b\} \leq c$. The second condition is always satisfied for sufficiently large r.

Now assume that $\gcd\{p, u\}$ equals p, u or 1. First consider the case where $u = pu'$. Then Eqs. (4.13) and (4.14) take the form

$$p2^q x + pu'2^v z - w = 0, \tag{4.20}$$

$$2^q y + u'2^v t = 0. \tag{4.21}$$

This system has a solution given by the matrix

$$M = \begin{bmatrix} 2^{r-q} & -u'2^v \\ 0 & 2^q \end{bmatrix}.$$

The point A is transformed into the point $(w, 0) = (p2^r, 0)$. The case when u divides p is similar.

Now let $\gcd\{p, u\} = 1$. Then there are integers i and j such that $pi + uj = 1$, whence $pi2^r + uj2^r = 2^r$, and consequently

$$p2^q i2^{r-q} + u2^v j2^{r-v} = 2^r, \tag{4.22}$$

so that one can take $i2^{r-q}$ as x and $j2^{r-v}$ as z. The system of Eqs. (4.13) and (4.14) has a solution given by the matrix

$$M = \begin{bmatrix} i2^{r-q} & -u2^v \\ j2^{r-v} & p2^q \end{bmatrix}.$$

The point A is transformed into the point $(w, 0) = (2^r, 0)$. □

5. The Type of a Dyadic Triangle

A dyadic triangle may be translated to an isomorphic triangle with the origin at one vertex.

Lemma 5.1. *Each dyadic triangle is isomorphic to a triangle ABC in which the vertex A lies at the origin, while both B and C have integral coordinates.*

Proof. First translate the given triangle to a triangle $AB'C'$ with A at the origin. Then transform $AB'C'$ to ABC using an (invertible) matrix of the form

$$\begin{bmatrix} 2^k & 0 \\ 0 & 2^l \end{bmatrix} \tag{5.1}$$

for sufficiently large k and l. □

The triangle in question may then be isomorphically transformed to a triangle in which the two remaining vertices are located in the first quadrant.

Lemma 5.2. *Each dyadic triangle is isomorphic to a triangle ABC contained in the first quadrant, with A located at the origin. Moreover, the vertices B and C may be chosen so that they have integral coordinates.*

Proof. Suppose that the given triangle does not already lie in the first quadrant. Then using a translation, or a translation and a reflection in one of the coordinate axes or in the line $y = x$ or $y = -x$ (cf. Lemma 4.4), one may transform it to a triangle with one vertex, call it A, at the origin, and the other two vertices, say $B_1 = (r, s)$ and $C_1 = (t, w)$, with non-negative second coordinates. We may assume that $0 < s \leq w$. If the triangle AB_1C_1 is in the first quadrant, then $B = B_1$ and $C = C_1$. If not, then we use a matrix

$$\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}, \tag{5.2}$$

where d is some positive integer such that $r + sd$ and $t + wd$ are non-negative, and at least one of them is positive. In this case $B = (r + sd, s)$ and $C = (t + wd, w)$. By Lemma 5.1, the triangle ABC is isomorphic to a triangle having vertices with integral coordinates, one located at the origin. □

We now show that a dyadic triangle ABC that is located in the first quadrant, with the origin at A, belongs to one of three basic types:

- the first type consists of the triangles of right type (as described in Definition 4.2);
- the second type comprises triangles, isomorphic to triangles with the vertex B on the x-axis, which are not of the first type;
- the third type comprises triangles in which neither B nor C is axial.

These three types may then be used to classify general dyadic triangles.

Definition 5.3. A dyadic triangle is defined to be of the *first*, *second*, or *third* (*basic*) *type* precisely when it is isomorphic to a first-quadrant triangle ABC (with A at the origin) of the corresponding type.

In particular, the first basic type corresponds to the right type of Definition 4.2.

Proposition 5.4. *Each triangle ABC with the origin as the vertex A and an axial vertex B is isomorphic to a triangle $AB'C'$ in the first quadrant such that the image B' of B is on any one of the coordinate axes, and both the vertices B' and C' have integral coordinates.*

Proof. Let B be axial. By Lemma 4.11, there is an automorphism of the plane which takes the triangle ABC to a triangle AB_1C_1 with B_1 on one of the axes. By Lemma 5.1, we may assume that the vertices have integral coordinates. Using reflections as in Lemma 4.4, the triangle AB_1C_1 may then be transformed to an isomorphic triangle $AB'C'$ in the first quadrant, with B' on any one of the coordinate axes. □

By reflection in the line $y = x$, the triangle $AB'C'$ of Proposition 5.4 may be transformed to a triangle with two vertices on the x-axis.

Proposition 5.5. *Let ABC be a triangle with $A = (0, 0)$, $B = (i, 0)$ and $C = (p, q)$ for some positive integers i, p, q with $p \neq i$. Then ABC is isomorphic to a triangle $T_{m,n}$ for some odd positive integers m and n if and only if $p = qp'$ or $q = p2^w$ or $q = 2^s$.*

Proof. First note that if a matrix

$$M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

transposes the triangle ABC into an isomorphic triangle $AB'C'$ with $B' = (m, 0)$, then necessarily $(ix, iy) = (m, 0)$, whence $y = 0$. Hence $\det M = xt = 2^r$, which implies $x = 2^u$ and $t = 2^v$ for some integers r, u, v, and

$$M = \begin{bmatrix} 2^u & 0 \\ z & 2^v \end{bmatrix}.$$

Moreover

$$(p, q)M = (px + qz, qt) = (p2^u + qz, q2^v) = (0, n)$$

precisely when

$$n = q2^v \quad \text{and} \quad p2^u + qz = 0\,. \tag{5.3}$$

The latter equation has a dyadic solution $z = -p2^u/q$ precisely when $p = qp'$ or $q = p2^w$ or $q = 2^s$ for some integers w and s. ☐

A triangle with one side parallel to one of the coordinate axes may be translated to a triangle with one side on this axis. It may then be transformed further, using translations and reflections as in Lemma 4.4, to a triangle in the first quadrant with one vertex at the origin and one side on the x-axis. Two typical situations are presented in Fig. 1.

Note that a triangle as in Fig. 1 may be further translated so that two vertices are on one axis and the third on the other. Now let $T_{m,n,k}$ denote the triangle ABC with vertices $A = (-k, 0)$, $B = (m, 0)$ and $C = (0, n)$ for some positive integers m, n and k.

Lemma 5.6. *Each triangle with one side parallel to one of the axes is isomorphic either to a triangle $T_{m,n,k}$, or to a triangle $T_{m,n}$.*

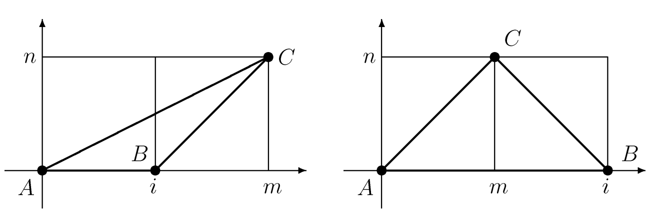

Fig. 1.

Proof. If a given triangle is not isomorphic to some $T_{m,n}$, then suppose that it may be translated to the triangle ABC presented in Fig. 1 with vertices $A = (0, 0)$, $B = (i, 0)$ and $C = (m, n)$, where $0 < i < m$. Transform the triangle using the matrix $\begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix}$, where d is a positive dyadic number. The transformation leaves the points A and B fixed, and takes C to $C' = (m - nd, n)$. If $m - nd$ is 0 or i, the triangle ABC' is a right triangle. Let $0 < m - nd < i$. This holds precisely when $(m - i)/n < d < m/n$. Such a dyadic number d always exists, and we can choose it so that our transformation will give a triangle with C' having the first coordinate between 0 and i. Using an appropriate matrix of the form (4.1), and possibly a translation, we can transform the latter triangle to a triangle of the form $T_{m',n',k'}$ for some positive integers m', n' and k'. □

The triangles considered in Lemma 5.6 are of the second or first basic types. Note that a triangle $T_{m,n,k}$ is isomorphic to a triangle $T_{m',n',k'}$, with n' and $\gcd\{m', k'\}$ odd. Note also that the vertices of the triangles $T_{m,n}$ and $T_{m,n,k}$ are axial. The results of this section combine to yield the following theorem.

Theorem 5.7. *Each dyadic triangle belongs to just one of the first, second or third basic types.*

6. Invariants of Dyadic Triangles

Section 3 showed that there is a bijective correspondence between isomorphism classes of dyadic intervals and odd positive integers. In other words, odd positive integers form complete invariants for the classification of dyadic intervals. In this section, we will show that certain quadruples of positive integers form complete invariants for classifying dyadic triangles up to isomorphism.

Definition 6.1. A dyadic triangle ABC is said to be *in special position* if it satisfies the following conditions:

(1) it lies in the first quadrant;
(2) A coincides with the origin;
(3) $B = (i, j)$ and $C = (m, n)$ are non-axial;
(4) the coordinates i, m, j, n of B and C are positive integers;
(5) neither $\gcd\{i, m\}$ or $\gcd\{j, n\}$ is even;
(6) $i < m$ and $j > n$.

Let $T_{i,m,j,n}$ denote a triangle ABC in special position.

Proposition 6.2. *Each dyadic triangle belonging to the third basic type is isomorphic to a triangle in special position.*

Proof. Lemma 5.2 shows that each dyadic triangle of third type may be transformed to an isomorphic triangle in one of the two positions depicted in Fig. 2,

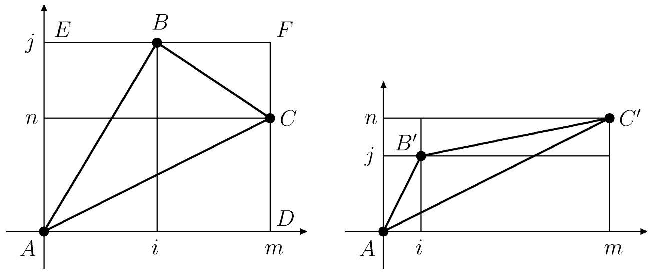

Fig. 2.

where i, j, m, n are positive integers. Note that in the triangle ABC we have $i < m$ and $j > n$, while in the triangle $AB'C'$ we have $i < m$ but $j < n$.

We will show that the triangle $AB'C'$ presented on the right-hand side of Fig. 2 is isomorphic to a triangle ABC of the kind presented on the left-hand side. First note that for any dyadic number d, the matrix $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ transposes the point B' to the point $B = (i, j + id)$ and the point C' to the point $C = (m, n + md)$, leaving A at the origin. We will show that d can be chosen in such a way that $j + id > n + md > 0$. The first inequality holds precisely when $d < (n - j)/(i - m)$. Since $n - j > 0$ and $i - m < 0$, the number d must be negative. Let $d' = -d$. Note that $j + id > 0$ and $n + md > 0$ precisely when $d' < \min\{j/i, n/m\}$. So for any d' satisfying this inequality, we may take $d = -d'$. Then if necessary, we may use an appropriate matrix from (4.1) to obtain an isomorphic triangle ABC with the required properties. □

Lemma 6.3. *Each dyadic triangle T is isomorphic to a triangle ABC with $A = (0, 0)$, $B = (i, j)$ and $C = (m, n)$.*

(1) *If T is of the first basic type, then $i = 0$, $n = 0$ and $j \leq m$. Moreover, ABC coincides with the triangle $T_{m,j}$.*
(2) *If T is of the second basic type, then $n = 0$, both $\gcd\{i, m\}$ and $\gcd\{j, 0\}$ are odd, and $i \leq m/2$. Moreover, ABC coincides with the triangle $T^s_{i,m,j}$ obtained from $T_{m-i,j,i}$ by translating it by the vector $(i, 0)$.*
(3) *If T is of the third basic type, then ABC coincides with the triangle $T_{i,m,j,n}$ in special position with $j \leq m$.*

Proof. The first statement follows by the fact that the triangles $T_{m,j}$ and $T_{j,m}$ are isomorphic. For the second, note that the triangles $T_{m-i,j,i}$ and $T_{i,j,m-i}$ are isomorphic. For the third, note that the triangles ABC and $AB'C'$, with $B' = (j, i)$ and $C' = (n, m)$, are isomorphic. □

Definition 6.4. The triangles $T_{m,j}$, $T^s_{i,m,j}$ and $T_{i,m,j,n}$ of Lemma 6.3 are called *representative triangles.*

Definition 6.5. A quadruple (i, m, j, n) of non-negative integers is said to be an *encoding quadruple* if it satisfies the following conditions:

(1) $0 \leq i < m$ and $0 \leq n < j$;
(2) $\gcd\{i, m\}$ and $\gcd\{j, n\}$ are odd;
(3) if i and n are both zero or both non-zero, then $j \leq m$;
(4) if just one of i and n is zero, then it is n, and $i \leq m/2$;
(5) if none of i, m, j, n is zero, then $i < m$ and $j > n$.

Denote the set of all encoding quadruples by $\mathcal{EQ}$. To each dyadic triangle T, we assign a member of $\mathcal{EQ}$ in the following way. First we transform T to an isomorphic representative triangle $T' = ABC$ of the appropriate basic type. Then we use Lemma 6.3 to assign an encoding quadruple as follows.

(1) If T' is representative of the first basic type, it coincides with some $T_{m,j}$ with $j \leq m$. In this case we assign the quadruple $(0, m, j, 0)$ to T.
(2) If T' is representative of the second basic type, then it coincides with some $T^s_{i,m,j}$. In this case we assign the quadruple $(i, m, j, 0)$ to T.
(3) Finally, if T' is representative of the third basic type, then it coincides with some $T_{i,m,j,n}$. In this case the corresponding quadruple is (i, m, j, n).

Note that each assigned quadruple belongs to $\mathcal{EQ}$.

Theorem 6.6. *Two dyadic triangles are isomorphic if and only if they have the same encoding quadruples.*

Proof. Isomorphic triangles are isomorphic to the same representative triangle, while no two distinct representative triangles are isomorphic. A representative triangle $T_{m,j}$ of the first basic type is determined by its right type $r_{m,j}$. A representative triangle $T^s_{i,m,j}$ of the second basic type is determined by the right types of the (right) triangles ABB' and $BB'C$, where $B' = (i, 0)$. A representative triangle $T_{i,m,j,n}$ of the third basic type is determined by the right types of the (right) triangles of Fig. 2, namely ABE with $E = (0, j)$ and ACD with $D = (m, 0)$. This establishes a one-to-one correspondence between representative triangles and encoding quadruples. □

7. Finite Generation

In this final section, it will be shown that dyadic polygons are finitely generated as algebras. The main task is to show that all triangles are finitely generated. We will first prove that the right triangles $T_{m,n}$ are finitely generated. Note that any set of generators of $T_{m,n}$ must contain the generators of its sides. We start by showing that the right triangles $T_{m,m}$ and $T_{m,1}$ are finitely generated.

Lemma 7.1. *Each right triangle of right type $r_{n,n}$ is finitely generated.*

Proof. Let ABC be a triangle of type $r_{n,n}$ with the origin as the vertex A, and with $B = (m, 0)$ and $C = (0, m)$ for some positive integer m. Note that each point $(k, 0)$, where k is an integer between 0 and m, is generated by (three) generators of the side AB. Similarly, each point $(0, k)$ is generated by (three) generators of the side AC. By Theorem 4.3, each line segment joining $(k, 0)$ and $(0, k)$ is of type k, if k is odd, and of type k' in the case $k = k'2^j$ for an odd k'.

The proof goes by induction on m. For $m = 1$, we have the triangle $T_{1,1}$ isomorphic to S_2, generated by its vertices. Now suppose that the proposition holds for a positive integer m, i.e., the triangle ABC is finitely generated. Note that all points of BC, in particular the points $B_k = (k, m-k)$, for $k = 0, 1, \dots, m$, are generated by the generators of BC. Then consider the triangle $AB'C'$, where $B' = (m+1, 0)$ and $C' = (0, m+1)$. We want to show that all the points of $AB'C'$ between the sides BC and $B'C'$ are generated by the points of BC and the points of $B'C'$. Let $C_k = (k+1, m-k)$. The points C_k are generated by the generators of $B'C'$. Note that each triangle $B_kC_kC_{k-1}$, and similarly each triangle $B_kC_kB_{k+1}$, is of right type $r_{1,1}$. Hence it is generated by its vertices. It follows that the triangle $AB'C'$ is generated by the finite union of the generating sets for ABC and $B'C'$. □

Corollary 7.2. *Each triangle $T_{m,m}$ is finitely generated.*

If a dyadic triangle ABC has a dyadic triangulation into subtriangles isomorphic to some T_{m_i,n_i}, where $i = 1, \dots, j$, then we say that it is a *triangulated sum* of these subtriangles, and write

$$ABC \cong T_{m_1,n_1} + \cdots + T_{m_j,n_j}. \tag{7.1}$$

A finite triangulated sum of finitely generated subtriangles is also finitely generated. In what follows, we will need the following technical lemma.

Lemma 7.3. *Let m and n be positive odd natural numbers. For each natural number i, there is an isomorphism between the triangle $\tilde{T}_{m,n}$, and the triangle $\tilde{T}^i_{m,n}$ with vertices $(0, 0)$, (in, n) and $(m + in, n)$.*

Proof. The matrix

$$\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \tag{7.2}$$

gives the required isomorphism. □

In the following lemma we will abuse notation by writing $T_{nk,n}$ for a triangle defined similarly as the triangle $T_{m,n}$ with odd m, n, but admitting the possibility that nk may be even.

Lemma 7.4. *Each triangle $T_{nk,n}$, for an odd positive integer n and a positive integer k, is finitely generated.*

Proof. We already know that $T_{1,1}$ is finitely generated. Now let $n = 1$ and $k > 1$. By Lemma 4.4, $T_{k,1} \cong \tilde{T}_{k,1}$. Note that $\tilde{T}^i_{1,1}$ has the vertices $(0,0)$, $(i,1)$ and $(i+1,1)$, where $i = 1, \ldots, k-1$. By Lemma 7.3, $\tilde{T}_{1,1} \cong T^i_{1,1}$ for each i. Moreover, $\tilde{T}_{k,1}$ is a triangulated sum of k copies of $T_{1,1}$. As each triangle $T_{1,1}$ is generated by its vertices, it follows that $T_{k,1}$ is generated by the generators of AB and the remaining vertex C.

Now let us fix an arbitrary positive odd integer n, and let $k > 1$. Consider the triangle $T_{kn,n}$. We may assume that its generators contain the standard generators of its sides. In particular, the generators of AB generate all the points $(i,0)$ for $i = 1, \ldots, kn-1$, and the generators of AC generate all the points $A_i = (0,i)$ for $i = 1, \ldots, n-1$. Note that the type of BC is n, and that the points $B = B_0 = C_0$, $C = A_n = C_n$ and $C_i = (k(n-i), i)$ for $i = 1, \ldots, n-1$ belong to the side BC, and generate it. Let $B_i = (k(n-i), 0)$ and $A = A_0 = B_n$. Now it is enough to observe that the line segments A_iC_i, B_iC_i and $A_{n-i}B_i$ provide a decomposition of ABC into a triangulated sum of right triangles isomorphic to $T_{n,1}$ or $\bar{T}_{n,1}$. Hence ABC may be considered as a finite triangulated sum of triangles $T_{n,1}$. As $T_{n,1}$ is finitely generated, it also follows that the triangle ABC is finitely generated. □

Corollary 7.5. *Each triangle $T_{m,1}$, for an odd positive integer $m > 1$, is generated by three generators of the side of type m together with the remaining vertex.*

Proof. This follows by the fact that $T_{m,1}$ is a triangulated sum of m copies of $T_{1,1}$. □

Proposition 7.6. *Each triangle $T_{m,n}$ is finitely generated.*

Proof. Without loss of generality assume that $n < m$. We will consider the triangle $\tilde{T}_{m,n}$ isomorphic to $T_{m,n}$, with the vertices $\tilde{A}, \tilde{B}, \tilde{C}$ defined as in Sec. 4. First note that by the Euclidean Algorithm:

$$\begin{aligned} m &= kn + m_1, \\ n &= k_1 m_1 + m_2, \\ m_1 &= k_2 m_2 + m_3, \\ &\vdots \\ m_{r-1} &= k_r m_r + m_{r+1}, \\ m_{r+1} &= k_{r+1} m_r. \end{aligned}$$

with $0 < m_i < m_{i-1}$. Let $A_0 = \tilde{A}, A_1 = (n,n), \ldots, A_k = (kn, n)$. Note that each triangle $\tilde{C}A_iA_{i+1}$ is isomorphic to $T_{n,n}$ — just use the matrix $\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}$. Similarly, the triangle $\tilde{C}A_kB$ is isomorphic to T_{n,m_1} — use the matrix $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$. Then $\tilde{A}\tilde{B}\tilde{C}$

is the triangulated sum of k triangles $\tilde{C}A_iA_{i+1}$, each isomorphic to $T_{n,n}$, and the triangle $\tilde{C}A_k\tilde{B}$ isomorphic to T_{n,m_1}.

We now repeat the above procedure with the triangle T_{n,m_1}, and decompose it as a triangulated sum of k_1 subtriangles isomorphic to T_{m_1,m_1} and a triangle isomorphic to T_{m_1,m_2}.

Continuing in the same way, after a finite number of steps, we end up with a triangle isomorphic to $T_{k_{r+1}m_r,m_r}$. By Lemma 7.4, this triangle is finitely generated. By Lemma 7.1, all the triangles T_{m_i,m_i} are finitely generated. Going back in the Euclidean Algorithm, we see that each triangle $T_{m_i,m_{i+1}}$ may be considered as the triangulated sum

$$k_iT_{m_i,m_i} + \cdots + k_rT_{m_r,m_r} + T_{k_{r+1}m_r,m_r},$$

where $k_jT_{m_j,m_j}$ denotes the triangulated sum of k_j copies of T_{m_j,m_j}. Finally, the triangle $\tilde{A}\tilde{B}\tilde{C}$, and hence also the triangle $T_{m,n}$, can be considered as the triangulated sum

$$kT_{n,n} + k_1T_{m_1,m_1} + k_2T_{m_2,m_2} + \cdots + k_rT_{m_r,m_r} + T_{k_{r+1}m_r,m_r}.$$

This means that $T_{m,n}$, as a finite triangulated sum of finitely generated subtriangles, is finitely generated. □

Corollary 7.7. *Each triangle with two vertices on one of the axes, and the third vertex on the other, is finitely generated.*

Proof. Such a triangle is either isomorphic to a triangle $T_{m,n}$, or is a triangulated sum of two triangles isomorphic to some T_{m_i,n_i}. □

Proposition 7.8. *If a dyadic triangle belongs to the third basic type, then it is finitely generated.*

Proof. Consider a dyadic triangle of third type. By Proposition 6.2, we may assume it to be isomorphic to a triangle ABC in special position, as in Fig. 2. We will show that the triangle ABC decomposes as a triangulated sum of certain subtriangles.

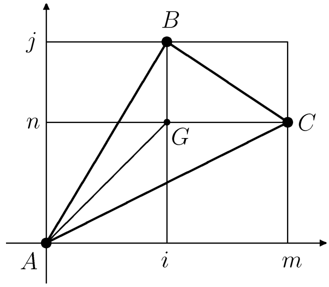

Fig. 3.

The first step is to decompose the triangle into subtriangles, each of which has one of its sides parallel to one of the coordinate axes (cf. Fig. 3). Write $G = (i, n)$. Then each of the triangles AGB, BGC, AGC has one side parallel to one of the coordinate axes. By Lemma 5.6, each of these triangles is isomorphic to a triangle of the form $T_{k,l}$ or $T_{p,q,r}$ for some positive integers p, q, r. Then each $T_{p,q,r}$ is a triangulated sum of two triangles isomorphic to triangles of the form $T_{k,l}$. It follows that ABC is a finite triangulated sum of finitely generated subtriangles. Hence it is finitely generated. □

Proposition 7.6 shows that triangles of first basic type are finitely generated. Corollary 7.7 shows that triangles of second basic type are finitely generated. Proposition 7.8 then completes the proof that each dyadic triangle is finitely generated. Now each dyadic polygon decomposes as a sum of dyadic triangles, as described in Sec. 2. We thus obtain the following theorem.

Theorem 7.9. *Each dyadic polygon is finitely generated.*

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